RELATIONS AND FUNCTIONS

- A relation R from a set A to a set B is a subset of the cartesian product $A \times B$ obtained by describing a relationship between the first element x and the second element y of the ordered pairs in $A \times B$.
- **Function :** A function f from a set A to a set B is a specific type of relation for which every element x of set A has one and only one image y in set B. We write $f: A \to B$, where $f(x) = y$.
- A function $f: X \to Y$ is one-one (or injective) if $f(x_1) = f(x_2) \Rightarrow x_1 = x_2 \ \forall \ x_1, x_2 \in X$.
- A function $f: X \to Y$ is onto (or surjective) if given any $y \in Y, \exists \ x \in X$ such that $f(x) = y$.
- **Many-One Function :** A function $f: A \to B$ is called many- one, if two or more different elements of A have the same f- image in B.
- **Into function :** A function $f: A \to B$ is into if there exist at least one element in B which is not the f - image of any element in A.
- **Many One -Onto function :** A function $f: A \to R$ is said to be many one- onto if f is onto but not one-one.
- **Many One -Into function :** A function is said to be many one-into if it is neither one-one nor onto.
- A function $f: X \to Y$ is invertible if and only if f is one-one and onto.

TRIGONOMETRIC FUNCTIONS AND EQUATIONS

- **General Solution of the equation $\sin\theta = 0$:**
  when $\sin\theta = 0$
  $\theta = n\pi : n \in I$ i.e. $n = 0, \pm 1, \pm 2$...........
  General solution of the equation $\cos\theta = 0$:
  when $\cos\theta = 0$
  $\theta = (2n+1)\pi/2, n \in I$. i.e. $n = 0, \pm 1, +2$.......

General solution of the equation $\tan\theta = 0$:
General solution of $\tan\theta = 0$ is $\theta = n\pi; n \in I$

- **General solution of the equation**
  (a) $\sin\theta = \sin\alpha$ **:** $\theta = n\pi + (-1)^n\alpha \ ; n \in I$
  (b) $\sin\theta = k$**, where** $-1 \le k \le 1$**.**
  $\theta = n\pi + (-1)^n\alpha$, where $n \in I$ and $\alpha = \sin^{-1}k$
  (c) $\cos\theta = \cos\alpha$ **:** $\theta = 2n\pi \pm \alpha, n \in I$
  (d) $\cos\theta = k$**, where** $-1 \le k \le 1$**.**
  $\theta = 2n\pi \pm \alpha$, where $n \in I$ and $\alpha = \cos^{-1}k$
  (e) $\tan\theta = \tan\alpha$ **:** $\theta = n\pi + \alpha \ ; n \in I$
  (f) $\tan\theta = k$**,** $\theta = n\pi + \alpha$, where $n \in I$ and $\alpha = \tan^{-1}k$
  (g) $\sin^2\theta = \sin^2\alpha$ **:** $\theta = n\pi \pm \alpha; n \in I$
  (h) $\cos^2\theta = \cos^2\alpha$ **:** $\theta = n\pi \pm \alpha \ ; n \in I$
  (i) $\tan^2\theta = \tan^2\alpha$ **:** $\theta = n\pi \pm \alpha \ ; n \in I$

- $\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \ldots\ldots$ to n terms
  $$= \frac{\sin\left[\alpha + \left(\frac{n-1}{2}\right)\beta\right]\left[\sin\left(\frac{n\beta}{2}\right)\right]}{\sin(\beta/2)} \ ; \ \beta \neq 2n\pi$$
- $\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \ldots\ldots$ to n terms
  $$= \frac{\cos\left[\alpha + \left(\frac{n-1}{2}\right)\beta\right]\left[\sin\left(\frac{n\beta}{2}\right)\right]}{\sin\left(\frac{\beta}{2}\right)} \ ; \ \beta \neq 2n\pi$$
- $\tan\left(\frac{B-C}{2}\right) = \left(\frac{b-c}{b+c}\right)\cot\left(\frac{A}{2}\right)$
- $\sin\left(\frac{A}{2}\right) = \sqrt{\frac{(s-b)(s-c)}{bc}}$
- $\tan\left(\frac{A}{2}\right) = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$
- $R = \frac{a}{2\sin A} = \frac{b}{2\sin B} = \frac{c}{2\sin C}$
- $R = \frac{abc}{4\Delta}$
- $r = 4R\sin\left(\frac{A}{2}\right).\sin\left(\frac{B}{2}\right).\sin\left(\frac{C}{2}\right)$
- $a = c\cos B + b\cos C$
- Maximum value of $a\sin\theta + b\cos\theta = \sqrt{a^2+b^2}$ and minimum value of $a\sin\theta + b\cos\theta = -\sqrt{a^2+b^2}$

INVERSE TRIGONOMETRIC FUNCTIONS

- **Properties of inverse trigonometric function**

• $\tan^{-1}x + \tan^{-1}y$

$$= \begin{cases} \tan^{-1}\left(\frac{x+y}{1-xy}\right), & \text{if } xy < 1 \\ \pi + \tan^{-1}\left(\frac{x+y}{1-xy}\right), & \text{if } x > 0, y > 0 \text{ and } xy > 1 \\ -\pi + \tan^{-1}\left(\frac{x+y}{1-xy}\right), & \text{if } x < 0, y < 0 \text{ and } xy > 1 \end{cases}$$

• $\tan^{-1}x - \tan^{-1}y$

$$= \begin{cases} \tan^{-1}\left(\frac{x-y}{1+xy}\right), & \text{if } xy > -1 \\ \pi + \tan^{-1}\left(\frac{x-y}{1+xy}\right), & \text{if } x > 0, y < 0 \text{ and } xy < -1 \\ -\pi + \tan^{-1}\left(\frac{x-y}{1+xy}\right), & \text{if } x < 0, y > 0 \text{ and } xy < -1 \end{cases}$$

• $\sin^{-1}x + \sin^{-1}y$

$$= \begin{cases} \sin^{-1}\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\}, & \text{if } -1 \le x,\ y \le 1 \text{ and } x^2+y^2 \le 1 \\ & \text{or if } xy < 0 \text{ and } x^2+y^2 > 1 \\ \pi - \sin^{-1}\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\}, & \text{if } 0 < x,\ y \le 1 \\ & \text{and } x^2+y^2 > 1 \\ -\pi - \sin^{-1}\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\}, & \text{if } -1 \le x,\ y < 0 \text{ and } x^2+y^2 > 1 \end{cases}$$

• $\cos^{-1}x + \cos^{-1}y$

$$= \begin{cases} \cos^{-1}\{xy - \sqrt{1-x^2}\sqrt{1-y^2}\}, & \text{if } -1 \le x, y \le 1 \text{ and } x+y \ge 0 \\ 2\pi - \cos^{-1}\{xy - \sqrt{1-x^2}\sqrt{1-y^2}\}, & \text{if } -1 \le x, y \le 1 \text{ and } x+y \le 0 \end{cases}$$

$$2\sin^{-1}x = \begin{cases} \sin^{-1}(2x\sqrt{1-x^2}), & \text{if } -\frac{1}{\sqrt{2}} \le x \le \frac{1}{\sqrt{2}} \\ \pi - \sin^{-1}(2x\sqrt{1-x^2}), & \text{if } \frac{1}{\sqrt{2}} \le x \le 1 \\ -\pi - \sin^{-1}(2x\sqrt{1-x^2}), & \text{if } -1 \le x \le -\frac{1}{\sqrt{2}} \end{cases}$$

$$2\tan^{-1}x = \begin{cases} \tan^{-1}\left(\frac{2x}{1-x^2}\right), & \text{if } -1 < x < 1 \\ \pi + \tan^{-1}\left(\frac{2x}{1-x^2}\right), & \text{if } x > 1 \\ -\pi + \tan^{-1}\left(\frac{2x}{1-x^2}\right), & \text{if } x < -1 \end{cases}$$

QUADRATIC EQUATIONS AND INEQUALITIES

Roots of a Quadratic Equation : The roots of the quadratic equation are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Nature of roots : In Quadratic equation $ax^2 + bx + c = 0$. The term $b^2 - 4ac$ is called discriminant of the equation. It is denoted by Δ or D.

(A) Suppose a, b, c $\in$ R and a $\neq$ 0

(i) If $D > 0 \Rightarrow$ Roots are Real and unequal

(ii) If $D = 0 \Rightarrow$ Roots are Real and equal and each equal to $-b/2a$

(iii) If $D < 0 \Rightarrow$ Roots are imaginary and unequal or complex conjugate.

(B) Suppose a, b, c $\in$ Q and a $\neq$ 0

(i) If $D > 0$ and D is perfect square $\Rightarrow$ Roots are unequal and Rational

(ii) If $D > 0$ and D is not perfect square $\Rightarrow$ Roots are irrational and unequal.

Condition for Common Root(s)

Let $ax^2 + bx + c = 0$ and $dx^2 + ex + f = 0$ have a common root α (say).

Condition for both the roots to be common is $\frac{a}{d} = \frac{b}{e} = \frac{c}{f}$

If $p + iq$ (p and q being real) is a root of the quadratic equation, where $i = \sqrt{-1}$, then $p - iq$ is also a root of the quadratic equation.

Every equation of n^{th} degree $(n \ge 1)$ has exactly n roots and if the equation has more than n roots, it is an identity.

COMPLEX NUMBERS

Exponential Form: If $z = x + iy$ is a complex number then its exponential form is $z = re^{i\theta}$ where r is modulus and θ is amplitude of complex number.

(i) $|z_1| + |z_2| \ge |z_1 + z_2|$; here equality holds when $\arg(z_1/z_2) = 0$ i.e. z_1 and z_2 are parallel.

(ii) $||z_1| - |z_2|| \le |z_1 - z_2|$; here equality holds when $\arg(z_1/z_2) = 0$ i.e. z_1 and z_2 are parallel.

(iii) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$

$\arg(z_1 z_2) = \theta_1 + \theta_2 = \arg(z_1) + \arg(z_2)$

$\arg\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2 = \arg(z_1) - \arg(z_2)$

For any integer k, $i^{4k} = 1$, $i^{4k+1} = i$, $i^{4k+2} = -1$, $i^{4k+3} = -i$

$|z - z_1| + |z - z_2| = \lambda$, represents an ellipse if $|z_1 - z_2| < \lambda$, having the points z_1 and z_2 as its foci. And if $|z_1 - z_2| = \lambda$, then z lies on a line segment connecting z_1 and z_2.

Properties of Cube Roots of Unity

(i) $1 + \omega + \omega^2 = 0$ (ii) $\omega^3 = 1$

(iii) $1 + \omega^n + \omega^{2n} = 3$ (if n is multiple of 3)

(iv) $1 + \omega^n + \omega^{2n} = 0$ (if n is not a multiple of 3).

PERMUTATIONS AND COMBINATIONS

The number of permutations of n different things, taken r at a time, where repetition is allowed, is n^r.

Selection of Objects with Repetition : The total number of selections of r things from n different things when each thing may be repeated any number of times is ${}^{n+r-1}C_r$

Selection from distinct objects :

The number of ways (or combinations) of n different things selecting at least one of them is ${}^nC_1 + {}^nC_2 + {}^nC_3 + \ldots + {}^nC_n = 2^n - 1$. This can also be stated as the total number of combination of n different things.

Selection from identical objects :

The number of ways to select some or all out of $(p + q + r)$ things where p are alike of first kind, q are alike of second kind and r are alike of third kind is $(p + 1)(q + 1)(r + 1) - 1$

Selection when both identical and distinct objects are present:

If out of $(p + q + r + t)$ things, p are alike one kind, q are alike of second kind, r are alike of third kind and t are different, then the total number of combinations is $(p + 1)(q + 1)(r + 1)2^t - 1$

Circular permutations:

(a) Arrangements round a circular table :

The number of circular permutations of n different things taken all at a time is $\frac{{}^nP_n}{n} = (n-1)!$, if clockwise and anticlockwise orders are taken as different.

(b) Arrangements of beads or flowers (all different) around a circular necklace or garland:
The number of circular permutations of 'n' different things taken all at a time is $\frac{1}{2}(n-1)!$, if clockwise and anticlockwise orders are taken to be some.

Sum of numbers :

(a) For given n different digits $a_1, a_2, a_3 ......a_n$ the sum of the digits in the unit place of all numbers formed (if numbers are not repeated) is $(a_1 + a_2 + a_3 + ...+ a_n)(n-1)!$

(b) Sum of the total numbers which can be formed with given n different digits $a_1, a_2, ...a_n$ is
$(a_1 + a_2 + a_3 + ...+ a_n)(n-1)!.(111 ....n \text{ times})$

BINOMIAL THEOREM

Greatest binomial coefficients : In a binomial expansion binomial coefficients of the middle terms are called as greatest binomial coefficients.

(a) If n is even : When $r = \frac{n}{2}$ i.e. ${}^nC_{n/2}$ takes maximum value.

(b) If n is odd : $r = \frac{n-1}{2}$ or $\frac{n+1}{2}$

i.e. ${}^nC_{\frac{n-1}{2}} = {}^nC_{\frac{n+1}{2}}$ and take maximum value.

Important Expansions :
If $|x| < 1$ and $n \in Q$ but $n \notin N$, then

(a) $$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + ...+ \frac{n(n-1).....(n-r+1)}{r!}x^r + ...$$

(b) $$(1-x)^n = 1 - nx + \frac{n(n-1)}{2!}x^2 - \frac{n(n-1)(n-2)}{3!}x^3 + ...+ \frac{n(n-1).....(n-r+1)}{r!}(-x)^r + ...$$

SEQUENCE AND SERIES

Properties related to A.P. :

(i) Common difference of AP is given by $d = S_2 - 2S_1$ where S_2 is sum of first two terms and S_1 is sum of first term.

(ii) If for an AP sum of p terms is q, sum of q terms is p, then sum of $(p + q)$ term is $(p + q)$.

(iii) In an A.P. the sum of terms equidistant from the beginning and end is constant and equal to sum of first and last terms.

(iv) If terms $a_1, a_2, ..., a_n, a_{n+1}, ..., a_{2n+1}$ are in A.P., then sum of these terms will be equal to $(2n + 1)a_{n+1}$.

(v) If for an A.P. sum of p terms is equal to sum of q terms then sum of $(p + q)$ terms is zero

(vi) Sum of n AM's inserted between a and b is equal to n times the single AM between a and b i.e. $\sum_{r=1}^{n} A_r = nA$ where $A = \frac{a+b}{2}$

The geometric mean (G.M.) of any two positive numbers a and b is given by $\sqrt{ab}$ i.e., the sequence a, G, b is G.P.

n GM's between two given numbers: If in between two numbers 'a' and 'b', we have to insert n GM $G_1, G_2, ... G_n$ then $a, G_1, G_2, ...G_n$, b will be in G.P.
The series consist of (n +2) terms and the last term is b and first term is a.

$$\Rightarrow ar^{n+2-1} = b \Rightarrow r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$$

$G_1 = ar, G_2 = ar^2 ....G_n = ar^n$ or $G_n = b/r$

Use of inequalities in progression :

(a) Arithmetic Mean $\geq$ Geometric Mean

(b) Geometric Mean $\geq$ Harmonic Mean :

$A \geq G \geq H$

STRAIGHT LINES

An acute angle (say θ) between lines L_1 and L_2 with slopes m_1 and m_2 is given by

$$\tan\theta = \left|\frac{m_2 - m_2}{1 + m_1m_2}\right|, \quad 1 + m_1m_2 \neq 0$$

Three points A, B and C are collinear, if and only if slope of AB = slope of BC.

The equation of the line having normal distance from origin is p and angle between normal and the positive x-axis is ω, is given by $x\cos\omega + y\sin\omega = p$.

Co-ordinate of some particular points :
Let $A(x_1,y_1)$, $B(x_2,y_2)$ and $C(x_3,y_3)$ are vertices of any triangle ABC, then

Incentre : Co-ordinates of incentre

$$\left(\frac{ax_1 + bx_2 + cx_3}{a+b+c}, \frac{ay_1 + by_2 + cy_3}{a+b+c}\right)$$

where a, b, c are the sides of triangle ABC

Area of a triangle : Let (x_1, y_1), (x_2, y_2) and (x_3, y_3) respectively be the coordinates of the vertices A, B, C of a triangle ABC. Then the area of triangle ABC, is

$$\frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$$

Or

$$= \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

CONIC SECTIONS

- **Condition of Tangency :** Circle $x^2 + y^2 = a^2$ will touch the line.

$y = mx + c$ if $c = \pm\ a\sqrt{1+m^2}$

- **Pair of Tangents :** From a given point $P(x_1, y_1)$ two tangents PQ and PR can be drawn to the circle $S = x^2 + y^2 + 2gx + 2fy + c = 0$. Their combined equation is $SS_1 = T^2$.
- **Condition of Orthogonality :** If the angle of intersection of the two circle is a right angle ($\theta = 90°$) then such circle are called Orthogonal circle and conditions for their orthogonality is $2g_1g_2 + 2f_1f_2 = c_1 + c_2$
- **Tangent to the parabola :**

Condition of Tangency : If the line $y = mx + c$ touches a parabola $y^2 = 4ax$ then $c = a/m$

- **Tangent to the Ellipse:**

Condition of tangency and point of contact :

The condition for the line $y = mx + c$ to be a tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is that $c^2 = a^2m^2 + b^2$ and the coordinates of the points of contact are $\left(\pm\frac{a^2m}{\sqrt{a^2m^2+b^2}}, \mp\frac{b^2}{\sqrt{a^2m^2+b^2}}\right)$

- **Normal to the ellipse**

(i) Point Form : The equation of the normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point (x_1, y_1) is $\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2$

(ii) Parametric Form : The equation of the normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point $(a\cos\theta, b\sin\theta)$ is $ax\sec\theta - by\,\text{cosec}\theta = a^2 - b^2$

- **Tangent to the hyperbola :**

Condition for tangency and points of contact : The condition for the line $y = mx + c$ to be a tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is that $c^2 = a^2m^2 - b^2$ and the coordinates of the points of contact are $\left(\pm\frac{a^2m}{\sqrt{a^2m^2-b^2}}, \pm\frac{b^2}{\sqrt{a^2m^2-b^2}}\right)$

- **Chord of contact :**

The equation of chord of contact of tangent drawn from a point $P(x_1, y_1)$ to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $T = 0$

where $T \equiv \frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1$

- **Equation of normal in different forms :**

Point Form : The equation of the normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point (x_1, y_1) is $\frac{a^2x}{x_1} + \frac{b^2y}{y_1} = a^2 + b^2$

- **Slope Form :** The equation of normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ in terms of slope 'm' is

$$y = mx \pm \frac{m(a^2+b^2)}{\sqrt{a^2 - b^2m^2}}$$

THREE DIMENSIONAL GEOMETRY

- **Conditions of Parallelism and Perpendicularity of Two Lines:**

Case-I : When dc's of two lines AB and CD, say ℓ_1, m_1, n_1 and ℓ_2, m_2, n_2 are known.

$AB \,||\, CD \Leftrightarrow \ell_1 = \ell_2, m_1 = m_2, n_1 = n_2$

$AB \perp CD \Leftrightarrow \ell_1\ell_2 + m_1m_2 + n_1n_2 = 0$

Case-II : When dr's of two lines AB and CD, say a_1, b_1, c_1 and a_2, b_2, c_2 are known

$$AB \,||\, CD \Leftrightarrow \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

$$AB \perp CD \Leftrightarrow a_1a_2 + b_1b_2 + c_1c_2 = 0$$

- If ℓ_1, m_1, n_1 and ℓ_2, m_2, n_2 are the direction cosines of two lines; and θ is the acute angle between the two lines; then $\cos\theta = |\ell_1\ell_2 + m_1m_2 + n_1n_2|$.
- Equation of a line through a point (x_1, y_1, z_1) and having direction cosines ℓ, m, n is $\frac{x-x_1}{\ell} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$
- Shortest distance between $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$ is $\left|\frac{(\vec{b}_1 \times \vec{b}_2).(\vec{a}_2 - \vec{a}_1)}{|\vec{b}_1 \times \vec{b}_2|}\right|$
- Let the two lines be

$$\frac{x-\alpha_1}{\ell_1} = \frac{y-\beta_1}{m_1} = \frac{z-\gamma_1}{n_1} \quad ....(1)$$

and $$\frac{x-\alpha_2}{\ell_2} = \frac{y-\beta_2}{m_2} = \frac{z-\gamma_2}{n_2} \quad ....(2)$$

These lines will coplanar if

$$\begin{vmatrix} \alpha_2-\alpha_1 & \beta_2-\beta_1 & \gamma_2-\gamma_1 \\ \ell_1 & m_1 & n_1 \\ \ell_2 & m_2 & n_2 \end{vmatrix} = 0$$

The plane containing the two lines is

$$\begin{vmatrix} x-\alpha_1 & y-\beta_1 & z-\gamma_1 \\ \ell_1 & m_1 & n_1 \\ \ell_2 & m_2 & n_2 \end{vmatrix} = 0$$

- The equation of a plane through a point whose position vector is $\vec{a}$ and perpendicular to the vector $\vec{N}$ is $(\vec{r} - \vec{a}).\vec{N} = 0$

- Vector equation of a plane that passes through the intersection of planes $\vec{r}.\vec{n}_1 = d_1$ and $\vec{r}.\vec{n}_2 = d_2$ is $\vec{r}.(\vec{n}_1 + \lambda\vec{n}_2) = d_1 + \lambda d_2$, where λ is any nonzero constant.
- Two planes $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$ are coplanar if $(\vec{a}_2 - \vec{a}_1) + (\vec{b}_1 \times \vec{b}_2) = 0$

DIFFERENTIAL CALCULUS

LIMIT

- **Existence of Limit :**

$\lim_{x\to a} f(x)$ exists $\Rightarrow \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = \ell$

Where ℓ is called the limit of the function

- (i) If $f(x) \le g(x)$ for every x in the deleted nbd of a, then $\lim_{x\to a} f(x) \le \lim_{x\to a} g(x)$

(ii) If $f(x) \le g(x) \le h(x)$ for every x in the deleted nbd of a and $\lim_{x\to a} f(x) = \ell = \lim_{x\to a} h(x)$ then $\lim_{x\to a} g(x) = \ell$

(iii) $\lim_{x\to a} fog(x) = f\left(\lim_{x\to a} g(x)\right) = f(m)$ where $\lim_{x\to a} g(x) = m$

(iv) If $\lim_{x\to a} f(x) = +\infty$ or $-\infty$, then $\lim_{x\to a} \frac{1}{f(x)} = 0$

CONTINUITY AND DIFFERENTIABILITY OF FUNCTIONS

- A function f(x) is said to be continuous at a point x = a if

$\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = f(a)$

- **Discontinuous Functions :**

(a) Removable Discontinuity:
A function f is said to have removable discontinuity at x = a if $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$ but their common value is not equal to f (a).

(b) Discontinuity of the first kind: A function f is said to have a discontinuity of the first kind at x = a if $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ both exist but are not equal.

(c) Discontinuity of second kind: A function f is said to have a discontinuity of the second kind at x = a if neither $\lim_{x\to a^-} f(x)$ nor $\lim_{x\to a^+} f(x)$ exists.

Similarly, if $\lim_{x\to a^+} f(x)$ does not exist, then f is said to have discontinuity of the second kind from the right at x = a.

- **For a function f :**

Differentiability $\Rightarrow$ Continuity;
Continuity $\not\Rightarrow$ derivability
Not derivibaility $\not\Rightarrow$ discontinuous ;
But discontinuity $\Rightarrow$ Non derivability

- **Differentiation of infinite series:**

(i) If $y = \sqrt{f(x) + \sqrt{f(x) + \sqrt{f(x) + ...\infty}}}$

$\Rightarrow \quad y = \sqrt{f(x) + y} \Rightarrow y^2 = f(x) + y$

$2y \frac{dy}{dx} = f'(x) + \frac{dy}{dx} \qquad \therefore \frac{dy}{dx} = \frac{f'(x)}{2y - 1}$

(ii) If $y = f(x)^{f(x)^{f(x)^{\dots\infty}}}$ then $y = f(x)^y$.

$\therefore \log y = y \log [f(x)]$

$\frac{1}{y}\frac{dy}{dx} = \frac{y'.f'(x)}{f(x)} + \log f(x).\left(\frac{dy}{dx}\right)$

$\therefore \frac{dy}{dx} = \frac{y^2 f'(x)}{f(x)[1 - y \log f(x)]}$

(iii) If $y = f(x) + \cfrac{1}{f(x) + \cfrac{1}{f(x) + \cfrac{1}{f(x)}}}$ .... then

$\frac{dy}{dx} = \frac{y f'(x)}{2y - f(x)}$

DIFFERENTIATION AND APPLICATION

- **Interpretation of the Derivative :** If y = f (x) then, m = f ′ (a) is the slope of the tangent line to y = f (x) at x = a
- **Increasing/Decreasing :**

(i) If $f'(x) > 0$ for all x in an interval I then f (x) is increasing on the interval I.

(ii) If $f'(x) < 0$ for all x in an interval I then f (x) is decreasing on the interval I.

(iii) If $f'(x) = 0$ for all x in an interval I then f (x) is constant on the interval I.

- **Test of Local Maxima and Minima –**

First Derivative Test – Let f be a differentiable function defined on an open interval I and $c \in I$ be any point. f has a local maxima or a local minima at x = c, f ′ (c) = 0.

Put $\frac{dy}{dx} = 0$ and solve this equation for x. Let $c_1, c_2........c_n$ be the roots of this.

If $\frac{dy}{dx}$ changes sign from +ve to –ve as x increases through c_1 then the function attains a local max at $x = c_1$

If $\frac{dy}{dx}$ changes its sign from –ve to +ve as x increases through c_1 then the function attains a local minimum at $x = c_1$

If $\frac{dy}{dx}$ does not changes sign as increases through c_1 then $x = c_1$ is neither a point of local max^m nor a point of local min^m. In this case x is a point of inflexion.

- **Rate of change of variable :**

The value of $\frac{dy}{dx}$ at $x = x_0$ i.e. $\left(\frac{dy}{dx}\right)_{x=x_0}$ represents the rate of change of y with respect to x at $x = x_0$

If $x = \phi(t)$ and $y = \psi(t)$, then $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$, provided that $\frac{dx}{dt} \neq 0$

Thus, the rate of change of y with respect to x can be calculated by using the rate of change of y and that of x each with respect to t.

- Length of Sub–tangent $= \left|y\frac{dx}{dy}\right|$; Sub–normal $= \left|y\frac{dy}{dx}\right|$;

Length of tangent $= \left|y\sqrt{\left\{1+\left(\frac{dx}{dy}\right)^2\right\}}\right|$

Length of normal $= \left|y\sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}\right|$

- **Equations of tangent and normal :** The equation of the tangent at P (x_1, y_1) to the curve $y = f(x)$ is

$$y - y_1 = \left(\frac{dy}{dx}\right)_P (x - x_1)$$

The equation of the normal at P (x_1, y_1) to the curve $y = f(x)$ is

$$y - y_1 = -\frac{1}{\left(\frac{dy}{dx}\right)_P}(x - x_1)$$

INTEGRAL CALCULUS

- **Two standard forms of integral :**

$$\int e^x\,[f(x) + f'(x)]\,dx = e^x f(x) + c$$

$$\Rightarrow \int e^x\,[f(x)+f'(x)]\,dx = \int e^x\,f(x)dx + \int e^x\,f'(x)dx$$

$$= e^x f(x) - \int e^x\,f'(x)\,dx + \int e^x\,f'(x)$$

(on integrating by parts) $= e^x f(x) + c$

- Table shows the partial fractions corresponding to different type of rational functions :

S. No.	Form of rational function	Form of partial fraction
1.	$\frac{px+q}{(x-a)(x-b)}$	$\frac{A}{(x-a)} + \frac{B}{(x-b)}$
2.	$\frac{px^2+qx+r}{(x-a)^2(x-b)}$	$\frac{A}{(x-a)} + \frac{B}{(x-a)^2} + \frac{C}{(x-b)}$
3.	$\frac{px^2+qx+r}{(x-a)(x^2+bx+c)}$	$\frac{A}{(x-a)} + \frac{Bx+C}{x^2+bx+C}$

- Leibnitz rule : $\frac{d}{dx}\int_{f(x)}^{g(x)} F(t)\,dt = g'(x)F(g(x)) - f'(x)F(f(x))$

- If a series can be put in the form $\frac{1}{n}\sum_{r=0}^{r=n-1} f\left(\frac{r}{n}\right)$ or $\frac{1}{n}\sum_{r=1}^{r=n} f\left(\frac{r}{n}\right)$, then its limit as $n \to \infty$ is $\int_0^1 f(x)\,dx$

- Area between curves :

$$y = f(x) \Rightarrow A = \int_a^b [\text{upper function}] - [\text{lower function}]\,dx$$

and $x = f(y) \Rightarrow A = \int_c^d [\text{right function}] - [\text{left function}]\,dy$

If the curves intersect then the area of each portion must be found individually.

- **Symmetrical area :** If the curve is symmetrical about a coordinate axis (or a line or origin), then we find the area of one symmetrical portion and multiply it by the number of symmetrical portion to get the required area.

PROBABILITY

- **Probability of an event:** For a finite sample space with equally likely outcomes Probability of an event is

$P(A) = \frac{n(A)}{n(S)}$, where n (A) = number of elements in the set A, n (S) = number of elements in the set S.

- **Theorem of total probability :** Let $\{E_1, E_2, ..., E_n\}$ be a partition of a sample space and suppose that each of $E_1, E_2, ..., E_n$ has nonzero probability. Let A be any event associated with S, then $P(A) = P(E_1)\,P(A \mid E_1) + P(E_2)\,P(A \mid E_2) + ... + P(E_n)\,P(A \mid E_n)$

- **Bayes' theorem:** If $E_1, E_2, ..., E_n$ are events which constitute a partition of sample space S, i.e. $E_1, E_2, ..., E_n$ are pairwise disjoint and $E_1 \cup E_2 \cup ... \cup E_n = S$ and A be any event with nonzero probability, then

$$P(E_i \mid A) = \frac{P(E_i)\,P(A \mid E_i)}{\sum_{j=1}^{n} P(E_j)\,P(A \mid E_j)}$$

- Let X be a random variable whose possible values $x_1, x_2, x_3, ..., x_n$ occur with probabilities $p_1, p_2, p_3, ...\, p_n$ respectively.

The mean of X, denoted by μ, is the number $\sum_{i=1}^{n} x_i p_i$

The mean of a random variable X is also called the expectation of X, denoted by E (X).

Trials of a random experiment are called Bernoulli trials, if they satisfy the following conditions :

(a) There should be a finite number of trials. (b) The trials should be independent. (c) Each trial has exactly two outcomes : success or failure. (d) The probability of success remains the same in each trial.

For Binomial distribution B (n, p),

$P(X = x) = {}^nC_x q^{n-x} p^x, x = 0, 1,..., n \ (q = 1 - p)$

MATRICES

Properties of Transpose

(i) $(A^T)^T = A$
(ii) $(A \pm B)^T = A^T \pm B^T$
(iii) $(AB)^T = B^T A^T$ (iv) $(kA)^T = k(A)^T$
(v) $I^T = I$ (vi) tr (A) = tr $(A)^T$
(vii) $(A_1A_2A_3.....A_{n-1}A_n)^T = A_n^T A_{n-1}^T.....A_3^T A_2^T A_1^T$

Symmetric Matrix : A square matrix $A = [a_{ij}]$ is called symmetric matrix if

$a_{ij} = a_{ji}$ for all i, j or $A^T = A$

Skew-Symmetric Matrix : A square matrix $A = [a_{ij}]$ is called skew-symmetric matrix if

$a_{ij} = -a_{ji}$ for all i, j or $A^T = -A$

Also every square matrix A can be uniquely expressed as a sum of a symmetric and skew-symmetric matrix.

Differentiation of a matrix : If $A = \begin{bmatrix} f(x) & g(x) \\ h(x) & \ell(x) \end{bmatrix}$ then

$\frac{dA}{dx} = \begin{bmatrix} f'(x) & g'(x) \\ h'(x) & \ell'(x) \end{bmatrix}$ is a differentiation of Matrix A.

DETERMINANTS

Properties of adjoint matrix : If A, B are square matrices of order n and I_n is corresponding unit matrix, then

(i) $A(\text{adj. } A) = |A| I_n = (\text{adj } A) A$
(ii) $|\text{adj } A| = |A|^{n-1}$ (Thus A (adj A) is always a scalar matrix)
(iii) $\text{adj}(\text{adj } A) = |A|^{n-2} A$
(iv) $|\text{adj}(\text{adj } A)| = |A|^{(n-1)^2}$
(v) $\text{adj}(A^T) = (\text{adj } A)^T$
(vi) adj (AB) = (adj B) (adj A)
(vii) $\text{adj}(A^m) = (\text{adj } A)^m, m \in N$
(viii) $\text{adj}(kA) = k^{n-1}(\text{adj. } A), k \in R$
(ix) $\text{adj}(I_n) = I_n$

Properties of Inverse Matrix : Let A and B are two invertible matrices of the same order, then

(i) $(A^T)^{-1} = (A^{-1})^T$
(ii) $(AB)^{-1} = B^{-1}A^{-1}$
(iii) $(A^k)^{-1} = (A^{-1})^k, k \in N$
(iv) $\text{adj}(A^{-1}) = (\text{adj } A)^{-1}$
(v) $(A^{-1})^{-1} = A$
(vi) $|A^{-1}| = \frac{1}{|A|} = |A|^{-1}$
(vii) If $A = \text{diag}(a_1, a_2, ..., a_n)$, then $A^{-1} = \text{diag}(a_1^{-1}, a_2^{-1}, ...a_n^{-1})$
(viii) A is symmetric matrix $\Rightarrow A^{-1}$ is symmetric matrix.

Rank of a Matrix : A number r is said to be the rank of a $m \times n$ matrix A if

(a) Every square sub matrix of order (r + 1) or more is singular and (b) There exists at least one square submatrix of order r which is non-singular.

Thus, the rank of matrix is the order of the highest order non-singular sub matrix.

Using Crammer's rule of determinant we get

$$\frac{x}{\Delta_1} = \frac{y}{\Delta_2} = \frac{z}{\Delta_3} = \frac{1}{\Delta} \text{ i. e. } x = \frac{\Delta_1}{\Delta}, y = \frac{\Delta_2}{\Delta}, z = \frac{\Delta_3}{\Delta}$$

Case-I : If $\Delta \neq 0$

Then $x = \frac{\Delta_1}{\Delta}, y = \frac{\Delta_2}{\Delta}, z = \frac{\Delta_3}{\Delta}$

$\therefore$ The system is consistent and has unique solutions.

Case-II if $\Delta = 0$ and

(i) If at least one of $\Delta_1, \Delta_2, \Delta_3$ is not zero then the system of equations a inconsistent i.e. has no solution.
(ii) If $d_1 = d_2 = d_3 = 0$ or $\Delta_1, \Delta_2, \Delta_3$ are all zero then the system of equations has infinitely many solutions.

VECTOR ALGEBRA

Given vectors $x_1\vec{a} + y_1\vec{b} + z_1\vec{c}$, $x_2\vec{a} + y_2\vec{b} + z_2\vec{c}$, $x_3\vec{a} + y_3\vec{b} + z_3\vec{c}$, where $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar vectors, will be coplanar if and only if $\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = 0$

Scalar triple product :

(a) If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ and $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$ then

$$(\vec{a} \times \vec{b}).\vec{c} = [\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

(b) [a b c] = volume of the parallelopiped whose coterminous edges are formed by $\vec{a}, \vec{b}, \vec{c}$

(c) $\vec{a}, \vec{b}, \vec{c}$ are coplanar if and only if $[\vec{a}\ \vec{b}\ \vec{c}] = 0$

(d) Four points A, B, C, D with position vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ respectively are coplanar if and only if $[\vec{AB}\ \vec{AC}\ \vec{AD}] = 0$ i.e. if and only if $[\vec{b} - \vec{a}\ \ \vec{c} - \vec{a}\ \ \vec{d} - \vec{a}] = 0$

(e) Volume of a tetrahedron with three coterminous edges $\vec{a}, \vec{b}, \vec{c} = \frac{1}{6}\left|[\vec{a}\ \vec{b}\ \vec{c}]\right|$

(f) Volume of prism on a triangular base with three coterminous edges $\vec{a}, \vec{b}, \vec{c} = \frac{1}{2}\left|[\vec{a}\ \vec{b}\ \vec{c}]\right|$

- **Lagrange's identity :**

$$(\vec{a}\times\vec{b}).(\vec{c}\times\vec{d}) = \begin{vmatrix} \vec{a}.\vec{c} & \vec{a}.\vec{d} \\ \vec{b}.\vec{c} & \vec{b}.\vec{d} \end{vmatrix} = (\vec{a}.\vec{c})(\vec{b}.\vec{d}) - (\vec{a}.\vec{d})(\vec{b}.\vec{c})$$

- **Reciprocal system of vectors :** If $\vec{a},\vec{b},\vec{c}$ be any three non coplanar vectors so that $[\vec{a}\ \vec{b}\ \vec{c}] \neq 0$ then the three vectors $\vec{a}'\vec{b}'\vec{c}'$ defined by the equations $\vec{a}' = \dfrac{\vec{b}\times\vec{c}}{[\vec{a}\ \vec{b}\ \vec{c}]}, \vec{b}' = \dfrac{\vec{c}\times\vec{a}}{[\vec{a}\ \vec{b}\ \vec{c}]}, \vec{c}' = \dfrac{\vec{a}\times\vec{b}}{[\vec{a}\ \vec{b}\ \vec{c}]}$ are called the reciprocal system of vectors to the given vectors $\vec{a},\vec{b},\vec{c}$.

STATISTICS

- **Relation between A.M., G.M. and H.M.**

A.M. $\geq$ G.M. $\geq$ H.M.

Equality sign holds only when all the observations in the series are same.

- **Relationship between mean, mode and median :**

(i) In symmetrical distribution

Mean = Mode = Median

(ii) In skew (moderately symmetrical) distribution

Mode = 3 median – 2 mean

- Mean deviation for ungrouped data

$$M.D.(\bar{x}) = \frac{\sum |x_i - \bar{x}|}{n}, \quad M.D.(M) = \frac{\sum |x_i - M|}{n}$$

- Mean deviation for grouped data

$$M.D.(\bar{x}) = \frac{\sum f_i |x_i - \bar{x}|}{N}, \quad M.D.(M) = \frac{\sum f_i |x_i - M|}{N},$$

where $N = \sum f_i$

- Variance and standard deviation for ungrouped data

$$\sigma^2 = \frac{1}{n}\sum (x_i - \bar{x})^2, \quad \sigma = \sqrt{\frac{1}{n}\sum (x_i - \bar{x})^2}$$

- Variance and standard deviation of a discrete frequency distribution

$$\sigma^2 = \frac{1}{n}\sum f_i (x_i - \bar{x})^2, \quad \sigma = \sqrt{\frac{1}{N}\sum f_i (x_i - \bar{x})^2}$$

- Variance and standard deviation of a continuous frequency distribution

$$\sigma^2 = \frac{1}{n}\sum f_i (x_i - \bar{x})^2, \quad \sigma = \sqrt{\frac{1}{N}\sum f_i x_i^2 - (\sum f_i x_i)^2}$$

- Coefficient of variation (C.V.) $= \dfrac{\sigma}{\bar{x}} \times 100, \ \bar{x} \neq 0$

For series with equal means, the series with lesser standard deviation is more consistent or less scattered.

DIFFERENTIAL EQUATIONS

- **Methods of solving a first order first degree differential equation :**

(a) Differential equation of the form $\dfrac{dy}{dx} = f(x)$

$$\frac{dy}{dx} = f(x) \Rightarrow dy = f(x)\,dx$$

Integrating both sides we obtain

$$\int dy = \int f(x)dx + c \quad \text{or } y = \int f(x)dx + c$$

(b) Differential equation of the form $\dfrac{dy}{dx} = f(x)\,g(y)$

$$\frac{dy}{dx} = f(x)\,g(y) \Rightarrow \int \frac{dy}{g(y)} = \int f(x)\,dx + c$$

(c) Differential equation of the form of $\dfrac{dy}{dx} = f(ax + by + c)$ **:**

To solve this type of differential equations, we put

$$ax + by + c = v \text{ and } \frac{dy}{dx} = \frac{1}{b}\left(\frac{dv}{dx} - a\right)$$

$$\therefore \frac{dv}{a + b f(v)} = dx$$

So solution is by integrating $\displaystyle\int \frac{dv}{a + b f(v)} = \int dx$

(d) Differential Equation of homogeneous type :

An equation in x and y is said to be homogeneous if it can be put in the form $\dfrac{dy}{dx} = \dfrac{f(x,y)}{g(x,y)}$ where f(x ,y) and g (x ,y) are both homogeneous functions of the same degree in x & y.

So to solve the homogeneous differential equation $\dfrac{dy}{dx} = \dfrac{f(x,y)}{g(x,y)}$, substitute y = vx and so $\dfrac{dy}{dx} = v + x\dfrac{dV}{dx}$

$$\text{Thus} \quad v + x\frac{dv}{dx} = f(v) \Rightarrow \frac{dx}{x} = \frac{dv}{f(v) - v}$$

Therefore solution is $\displaystyle\int \frac{dx}{x} = \int \frac{dv}{f(v) - v} + c$

- **Linear differential equations :**

$$\frac{dy}{dx} + Py = Q \qquad \text{.......(1)}$$

Where P and Q are either constants or functions of x.

Multiplying both sides of (1) by $e^{\int P\,dx}$, we get

$$e^{\int P\,dx}\left(\frac{dy}{dx} + Py\right) = Q\ e^{\int P\,dx}$$

On integrating both sides with respect to x we get

$$y\ e^{\int P\,dx} = \int Q\ e^{\int P\,dx} + c$$

which is the required solution, where c is the constant and $e^{\int P\,dx}$ is called the integration factor.