If $z = a + ib$ then its conjugate complex is obtained by changing the sign of its imaginary part & is denoted by $\bar{z}$. i.e. $\bar{z} = a - ib$.

Note that : www.MathsBySuhag.com , www.TekoClasses.com

(i) $z + \bar{z} = 2\,\text{Re}(z)$ **(ii)** $z - \bar{z} = 2i\,\text{Im}(z)$ **(iii)** $z\bar{z} = a^2 + b^2$ which is real

(iv) If z lies in the 1st quadrant then $\bar{z}$ lies in the 4th quadrant and $-\bar{z}$ lies in the 2nd quadrant.

3. ALGEBRAIC OPERATIONS :

The algebraic operations on complex numbers are similiar to those on real numbers treating i as a polynomial. Inequalities in complex numbers are not defined. There is no validity if we say that complex number is positive or negative.

e.g. $z > 0$, $4 + 2i < 2 + 4i$ are meaningless .

However in real numbers if $a^2 + b^2 = 0$ then $a = 0 = b$ but in complex numbers,

$z_1^2 + z_2^2 = 0$ does not imply $z_1 = z_2 = 0$.www.MathsBySuhag.com , www.TekoClasses.com

4. EQUALITY IN COMPLEX NUMBER :

Two complex numbers $z_1 = a_1 + ib_1$ & $z_2 = a_2 + ib_2$ are equal if and only if their real & imaginary parts coincide.

5. REPRESENTATION OF A COMPLEX NUMBER IN VARIOUS FORMS:

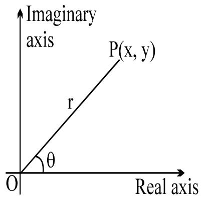
(a) Cartesian Form (Geometric Representation) :

Every complex number $z = x + i\,y$ can be represented by a point on the cartesian plane known as complex plane (Argand diagram) by the ordered pair (x, y).

length OP is called modulus of the complex number denoted by $|z|$ & θ is called the argument or amplitude

eg . $|z| = \sqrt{x^2 + y^2}$

$\theta = \tan^{-1}\frac{y}{x}$ (angle made by OP with positive x–axis)



NOTE :(i) $|z|$ is always non negative . Unlike real numbers $|z| = \begin{bmatrix} z & \text{if } z > 0 \\ -z & \text{if } z < 0 \end{bmatrix}$ is **not correct**

(ii) Argument of a complex number is a many valued function . If θ is the argument of a complex number then $2\,n\pi + \theta$; $n \in I$ will also be the argument of that complex number. Any two arguments of a complex number differ by $2n\pi$.

(iii) The unique value of θ such that $-\pi < \theta \le \pi$ is called the principal value of the argument.

(iv) Unless otherwise stated, amp z implies principal value of the argument.

(v) By specifying the modulus & argument a complex number is defined completely. For the complex number 0 + 0 i the argument is not defined and this is the only complex number which is given by its modulus.www.MathsBySuhag.com , www.TekoClasses.com

(b) $|z| \ge 0$; $|z| \ge \text{Re}(z)$; $|z| \ge \text{Im}(z)$; $|z| = |\bar{z}| = |-z|$; $z\bar{z} = |z|^2$;

$|z_1 z_2| = |z_1| \cdot |z_2|$; $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$, $z_2 \neq 0$, $|z^n| = |z|^n$;

$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\left[|z_1|^2 + |z_2|^2\right]$www.MathsBySuhag.com , www.TekoClasses.com

$\big||z_1| - |z_2|\big| \le |z_1 + z_2| \le |z_1| + |z_2|$ **[TRIANGLE INEQUALITY]**

(c) (i) $\text{amp}\,(z_1 . z_2) = \text{amp}\ z_1 + \text{amp}\ z_2 + 2\,k\pi$. $k \in I$

(ii) $\text{amp}\left(\frac{z_1}{z_2}\right) = \text{amp}\ z_1 - \text{amp}\ z_2 + 2\,k\pi$; $k \in I$

(iii) $\text{amp}(z^n) = n\ \text{amp}(z) + 2k\pi$.

where proper value of k must be chosen so that RHS lies in $(-\pi, \pi]$.

(7) VECTORIAL REPRESENTATION OF A COMPLEX :

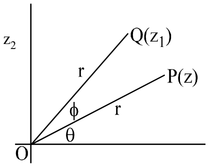
Every complex number can be considered as if it is the position vector of that point. If the point P represents the complex number z then, $\vec{OP} = z$ & $|\vec{OP}| = |z|$.

NOTE :(i) If $\vec{OP} = z = r\,e^{i\theta}$ then $\vec{OQ} = z_1 = r\,e^{i(\theta+\phi)} = z \,.\, e^{i\phi}$. If $\vec{OP}$ and $\vec{OQ}$ are of unequal magnitude then $\hat{OQ} = \hat{OP}\,e^{i\phi}$

(ii) If A, B, C & D are four points representing the complex numbers z_1, z_2, z_3 & z_4 then

AB || CD if $\frac{z_4 - z_3}{z_2 - z_1}$ is purely real ;

AB ⊥ CD if $\frac{z_4 - z_3}{z_2 - z_1}$ is purely imaginary]



(iii) If z_1, z_2, z_3 are the vertices of an equilateral triangle where z_0 is its circumcentre then **(a)** $z_1^2 + z_2^2 + z_3^2 - z_1 z_2 - z_2 z_3 - z_3 z_1 = 0$ **(b)** $z_1^2 + z_2^2 + z_3^2 = 3\,z_0^2$

8. DEMOIVRE'S THEOREM :

Statement : $\cos n\theta + i \sin n\theta$ is the value or one of the values of $(\cos\theta + i\sin\theta)^n$ ¥ $n \in Q$. The theorem is very useful in determining the roots of any complex quantity

Note : Continued product of the roots of a complex quantity should be determined using theory of equations.

(ii) $1^p + \alpha_1^p + \alpha_2^p + ... + \alpha_{n-1}^p = 0$ if p is not an integral multiple of n

$= n$ if p is an integral multiple of n

(iii) $(1-\alpha_1)(1-\alpha_2) ...... (1-\alpha_{n-1}) = n$ &

$(1+\alpha_1)(1+\alpha_2) .... (1+\alpha_{n-1}) = 0$ if n is even and 1 if n is odd.

(iv) $1 . \alpha_1 . \alpha_2 . \alpha_3 .... \alpha_{n-1} = 1$ or -1 according as n is odd or even.

11. THE SUM OF THE FOLLOWING SERIES SHOULD BE REMEMBERED :

(i) $\cos\theta + \cos 2\theta + \cos 3\theta + ... + \cos n\theta = \dfrac{\sin(n\theta/2)}{\sin(\theta/2)} \cos\left(\dfrac{n+1}{2}\right)\theta.$

(ii) $\sin\theta + \sin 2\theta + \sin 3\theta + ... + \sin n\theta = \dfrac{\sin(n\theta/2)}{\sin(\theta/2)} \sin\left(\dfrac{n+1}{2}\right)\theta.$

Note : If $\theta = (2\pi/n)$ then the sum of the above series vanishes.

12. STRAIGHT LINES & CIRCLES IN TERMS OF COMPLEX NUMBERS :

(A) If z_1 & z_2 are two complex numbers then the complex number $z = \dfrac{nz_1 + mz_2}{m+n}$ divides the joins of z_1 & z_2 in the ratio m : n.

Note:(i) If a , b , c are three real numbers such that $az_1 + bz_2 + cz_3 = 0$; where $a + b + c = 0$ and a,b,c are not all simultaneously zero, then the complex numbers z_1 , z_2 & z_3 are collinear.

(ii) If the vertices A, B, C of a Δ represent the complex nos. z_1, z_2, z_3 respectively, then :

(a) Centroid of the ΔABC $= \dfrac{z_1 + z_2 + z_3}{3}$: **(b)** Orthocentre of the ΔABC $=$

$$\frac{(a\sec A)z_1 + (b\sec B)z_2 + (c\sec C)z_3}{a\sec A + b\sec B + c\sec C} \quad \textbf{OR} \quad \frac{z_1\tan A + z_2\tan B + z_3\tan C}{\tan A + \tan B + \tan C}$$

(c) Incentre of the ΔABC $= (az_1 + bz_2 + cz_3) \div (a + b + c)$.

(d) Circumcentre of the ΔABC = :

$(Z_1 \sin 2A + Z_2 \sin 2B + Z_3 \sin 2C) \div (\sin 2A + \sin 2B + \sin 2C)$.

(B) $\text{amp}(z) = \theta$ is a ray emanating from the origin inclined at an angle θ to the x– axis.

(C) $|z - a| = |z - b|$ is the perpendicular bisector of the line joining a to b.

(D) The equation of a line joining z_1 & z_2 is given by ;

$z = z_1 + t\,(z_1 - z_2)$ where t is a perameter.www.MathsBySuhag.com , www.TekoClasses.com

(E) $z = z_1\,(1 + it)$ where t is a real parameter is a line through the point z_1 & perpendicular to oz_1.

(F) The equation of a line passing through z_1 & z_2 can be expressed in the determinant form as $\begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix}$

13.(a) Reflection points for a straight line : Two given points P & Q are the reflection points for a given straight line if the given line is the right bisector of the segment PQ. Note that the two points denoted by the complex numbers z_1 & z_2 will be the reflection points for the straight line $\bar{\alpha}z + \alpha\bar{z} + r = 0$ if and only if ; $\bar{\alpha}z_1 + \alpha\bar{z}_2 + r = 0$, where r is real and α is non zero complex constant.

(b) Inverse points w.r.t. a circle : www.MathsBySuhag.com , www.TekoClasses.com

Two points P & Q are said to be inverse w.r.t. a circle with centre 'O' and radius ρ, if :

(i) the point O, P, Q are collinear and on the same side of O. **(ii)** $OP . OQ = \rho^2$.

Note that the two points z_1 & z_2 will be the inverse points w.r.t. the circle

$z\bar{z} + \bar{\alpha}z + \alpha\bar{z} + r = 0$ if and only if $z_1\bar{z}_2 + \bar{\alpha}z_1 + \alpha\bar{z}_2 + r = 0$.

14. PTOLEMY'S THEOREM : www.MathsBySuhag.com , www.TekoClasses.com

It states that the product of the lengths of the diagonals of a convex quadrilateral inscribed in a circle is equal to the sum of the lengths of the two pairs of its opposite sides.

i.e. $|z_1 - z_3|\,|z_2 - z_4| = |z_1 - z_2|\,|z_3 - z_4| + |z_1 - z_4|\,|z_2 - z_3|$.

15. LOGARITHM OF A COMPLEX QUANTITY :

(i) $\text{Log}_e(\alpha + i\beta) = \dfrac{1}{2}\text{Log}_e(\alpha^2 + \beta^2) + i\left(2n\pi + \tan^{-1}\dfrac{\beta}{\alpha}\right)$ where $n \in I$.

(ii) i^i represents a set of positive real numbers given by $e^{-\left(2n\pi + \frac{\pi}{2}\right)}$, $n \in I$.

2.THEORY OF EQUATIONS (QUADRATIC EQUATIONS)

The general form of a quadratic equation in x is, $ax^2 + bx + c = 0$, where $a , b , c \in R$ & $a \neq 0$.

RESULTS :1. The solution of the quadratic equation , $ax^2 + bx + c = 0$ is given by $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

The expression $b^2 - 4ac = D$ is called the discriminant of the quadratic equation.

2. If α & β are the roots of the quadratic equation $ax^2 + bx + c = 0$, then;

(i) $\alpha + \beta = -b/a$ (ii) $\alpha\beta = c/a$ (iii) $\alpha - \beta = \sqrt{D}/a$.

3.NATURE OF ROOTS:(A) Consider the quadratic equation $ax^2 + bx + c = 0$ where $a, b, c \in R$ & $a \neq 0$ then **(i)** $D > 0 \Leftrightarrow$ roots are real & distinct (unequal). **(ii)** $D = 0 \Leftrightarrow$ roots are real & coincident (equal). **(iii)** $D < 0 \Leftrightarrow$ roots are imaginary . **(iv)** If $p + iq$ is one root of a quadratic equation, then the other must be the conjugate $p - iq$ & vice versa. $(p , q \in R$ & $i = \sqrt{-1})$.

(B) Consider the quadratic equation $ax^2 + bx + c = 0$ where $a, b, c \in Q$ & $a \neq 0$ then;

(i) If $D > 0$ & is a perfect square , then roots are rational & unequal.

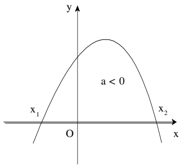
(ii) If $\alpha = p + \sqrt{q}$ is one root in this case, (where p is rational & $\sqrt{q}$ is a surd) then the other

Roots are real &

Roots are

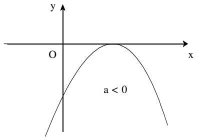
Roots are complex

Fig. 4

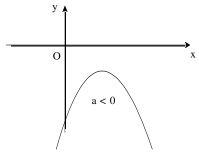


Roots are real &

Fig. 5



Roots are



Roots are complex

7. SOLUTION OF QUADRATIC INEQUALITIES:

$ax^2 + bx + c > 0$ $(a \neq 0)$.

(i) If $D > 0$, then the equation $ax^2 + bx + c = 0$ has two different roots $x_1 < x_2$.

Then $a > 0 \Rightarrow x \in (-\infty, x_1) \cup (x_2, \infty)$

$a < 0 \Rightarrow x \in (x_1, x_2)$www.MathsBySuhag.com , www.TekoClasses.com

(ii) If $D = 0$, then roots are equal, i.e. $x_1 = x_2$.

In that case $a > 0 \Rightarrow x \in (-\infty, x_1) \cup (x_1, \infty)$

$a < 0 \Rightarrow x \in \phi$

(iii) Inequalities of the form $\frac{P(x)}{Q(x)}$ 0 can be quickly solved using the method of intervals.

8.MAXIMUM & MINIMUM VALUE of $y = ax^2 + bx + c$ occurs at $x = -(b/2a)$ according as ; $a < 0$ or $a > 0$. $y \in \left[\frac{4ac - b^2}{4a}, \infty\right)$ if $a > 0$ & $y \in \left(-\infty, \frac{4ac - b^2}{4a}\right]$ if $a < 0$.

9.COMMON ROOTS OF 2 QUADRATIC EQUATIONS [ONLY ONE COMMON ROOT] :

Let α be the common root of $ax^2 + bx + c = 0$ & $a'x^2 + b'x + c' = 0$ Therefore $a\alpha^2 + b\alpha + c = 0$; $a'\alpha^2 + b'\alpha + c' = 0$. By Cramer's Rule $\frac{\alpha^2}{bc' - b'c} = \frac{\alpha}{a'c - ac'} = \frac{1}{ab' - a'b}$ Therefore, $\alpha = \frac{ca' - c'a}{ab' - a'b} = \frac{bc' - b'c}{a'c - ac'}$. So the condition for a common root is $(ca' - c'a)^2 = (ab' - a'b)(bc' - b'c)$.

10. The condition that a quadratic function $f(x, y) = ax^2 + 2hxy + by^2 + 2gx + 2fy + c$ may be resolved into two linear factors is that ;

signs, then $f(x) = 0$ must have atleast one real root between a and b .

(vi) Every eqtion $f(x) = 0$ of degree odd has atleast one real root of a sign opposite to that of its last term.

12. LOCATION OF ROOTS : www.MathsBySuhag.com , www.TekoClasses.com

Let $f(x) = ax^2 + bx + c$, where $a > 0$ & $a, b, c \in R$.

(i) Conditions for both the roots of $f(x) = 0$ to be greater than a specified number 'd' are $b^2 - 4ac \geq 0$; $f(d) > 0$ & $(-b/2a) > d$.

(ii) Conditions for both roots of $f(x) = 0$ to lie on either side of the number 'd' (in other words the number 'd' lies between the roots of $f(x) = 0$) is $f(d) < 0$.

(iii) Conditions for exactly one root of $f(x) = 0$ to lie in the interval (d, e) i.e. $d < x < e$ are $b^2 - 4ac > 0$ & $f(d) . f(e) < 0$.

(iv) Conditions that both roots of $f(x) = 0$ to be confined between the numbers p & q are $(p < q)$. $b^2 - 4ac \geq 0$; $f(p) > 0$; $f(q) > 0$ & $p < (-b/2a) < q$.

13. LOGARITHMIC INEQUALITIES

(i) For $a > 1$ the inequality $0 < x < y$ & $\log_a x < \log_a y$ are equivalent.

(ii) For $0 < a < 1$ the inequality $0 < x < y$ & $\log_a x > \log_a y$ are equivalent.

(iii) If $a > 1$ then $\log_a x < p \Rightarrow 0 < x < a^p$

(iv) If $a > 1$ then $\log_a x > p \Rightarrow x > a^p$

(v) If $0 < a < 1$ then $\log_a x < p \Rightarrow x > a^p$

(vi) If $0 < a < 1$ then $\log_a x > p \Rightarrow 0 < x < a^p$

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*3.*Sequence & Progression(AP, GP, HP, AGP, Spl. Series)

DEFINITION : A sequence is a set of terms in a definite order with a rule for obtaining the terms. e.g. 1 , 1/2 , 1/3 , ... , 1/n , ... is a sequence.

AN ARITHMETIC PROGRESSION (AP) :AP is a sequence whose terms increase or decrease by a fixed number. This fixed number is called the common difference. If a is the first term & d the common difference, then AP can be written as a, a + d, a + 2 d, ... a + (n – 1)d, ... n^{th} term of this AP $t_n = a + (n - 1)d$, where $d = a_n - a_{n-1}$. The sum of the first n terms of the AP is given by ; $S_n = \frac{n}{2}[2a + (n - 1)d] = \frac{n}{2}[a + l]$. where l is the last term.

NOTES :(i) If each term of an A.P. is increased, decreased, multiplied or divided by the same non zero number, then the resulting sequence is also an AP.

(ii) Three numbers in AP can be taken as a – d, a, a + d ; four numbers in AP can be taken as a – 3d, a – d, a + d, a + 3d ; five numbers in AP are a – 2d , a – d, a, a + d, a + 2d & six terms in AP are a – 5d, a – 3d, a – d, a + d, a + 3d, a + 5d etc.

(iii) The common difference can be zero, positive or negative.

(iv) The sum of the two terms of an AP equidistant from the beginning & end is constant and equal to the sum of first & last terms.

HARMONIC PROGRESSION (HP) : A sequence is said to HP if the reciprocals of its terms are in AP. If the sequence $a_1, a_2, a_3, \dots, a_n$ is an HP then $1/a_1, 1/a_2, \dots, 1/a_n$ is an AP & converse. Here we do not have the formula for the sum of the n terms of an HP. For HP whose first term is a & second term is b, the n^{th} term is $t_n = \dfrac{ab}{b + (n-1)(a-b)}$.

If a, b, c are in HP $\Rightarrow b = \dfrac{2ac}{a+c}$ or $\dfrac{a}{c} = \dfrac{a-b}{b-c}$.

MEANS

ARITHMETIC MEAN :

If three terms are in AP then the middle term is called the AM between the other two, so if a, b, c are in AP, b is AM of a & c . AM for any n positive number $a_1, a_2, \dots, a_n$ is ;

$A = \dfrac{a_1 + a_2 + a_3 + \dots + a_n}{n}$ .www.MathsBySuhag.com , www.TekoClasses.com

n - ARITHMETIC MEANS BETWEEN TWO NUMBERS :

If a, b are any two given numbers & a, $A_1, A_2, \dots, A_n$, b are in AP then $A_1, A_2, \dots A_n$ are the n AM's between a & b .

$A_1 = a + \dfrac{b-a}{n+1}$, $A_2 = a + \dfrac{2(b-a)}{n+1}$, ... , $A_n = a + \dfrac{n(b-a)}{n+1}$ $= a + d$, $= a + 2d$, ... , $A_n = a + nd$,

where $d = \dfrac{b-a}{n+1}$

Note : Sum of n AM's inserted between a & b is equal to n times the single AM between a & b i.e.

$\sum_{r=1}^{n} A_r = nA$ where A is the single AM between a & b.

GEOMETRIC MEANS :

If a, b, c are in GP, b is the GM between a & c.

$b^2 = ac$, therefore $b = \sqrt{ac}$; $a > 0$, $c > 0$.

n-GEOMETRIC MEANS BETWEEN a, b :

If a, b are two given numbers & a, $G_1, G_2, \dots, G_n$, b are in GP. Then $G_1, G_2, G_3, \dots, G_n$ are n GMs between a & b .

$G_1 = a(b/a)^{1/n+1}$, $G_2 = a(b/a)^{2/n+1}$, ... , $G_n = a(b/a)^{n/n+1}$
$= ar$, $= ar^2$, ... $= ar^n$, where $r = (b/a)^{1/n+1}$

Note : The product of n GMs between a & b is equal to the n^{th} power of the single GM between a & b

SIGMA NOTATIONS **THEOREMS :**

(i) $\sum_{r=1}^{n} (a_r \pm b_r) = \sum_{r=1}^{n} a_r \pm \sum_{r=1}^{n} b_r$. **(ii)** $\sum_{r=1}^{n} k\,a_r = k \sum_{r=1}^{n} a_r$. **(iii)** $\sum_{r=1}^{n} k = nk$; where k is a constant.

RESULTS **(i)** $\sum_{r=1}^{n} r = \dfrac{n(n+1)}{2}$ (sum of the first n natural nos.)

(ii) $\sum_{r=1}^{n} r^2 = \dfrac{n(n+1)(2n+1)}{6}$ (sum of the squares of the first n natural numbers)

(iii) $\sum_{r=1}^{n} r^3 = \dfrac{n^2(n+1)^2}{4} \left[\sum_{r=1}^{n} r\right]^2$ (sum of the cubes of the first n natural numbers)

(iv) $\sum_{r=1}^{n} r^4 = \dfrac{n}{30}(n+1)(2n+1)(3n^2+3n-1)$www.MathsBySuhag.com , www.TekoClasses.com

METHOD OF DIFFERENCE : If $T_1, T_2, T_3, \dots, T_n$ are the terms of a sequence then some times the terms $T_2 - T_1, T_3 - T_2, \dots$ constitute an AP/GP. n^{th} term of the series is determined & the sum to n terms of the sequence can easily be obtained.

Remember that to find the sum of n terms of a series each term of which is composed of r factors in AP, the first factors of several terms being in the same AP, we "write down the nth term, affix the next factor at the end, divide by the number of factors thus increased and by the common difference and add a constant. Determine the value of the constant by applying the initial conditions".

4.PERMUTATION AND COMBINATION

DEFINITIONS :1. PERMUTATION : Each of the arrangements in a definite order which can be made by taking some or all of a number of things is called a **PERMUTATION**.

2.COMBINATION : Each of the groups or selections which can be made by taking some or all of a number of things without reference to the order of the things in each group is called a **COMBINATION**.

FUNDAMENTAL PRINCIPLE OF COUNTING :

If an event can occur in 'm' different ways, following which another event can occur in '*n*' different ways, then the total number of different ways of simultaneous occurrence of both events in a definite order is $m \times n$. This can be extended to any number of events.

RESULTS :(i) A Useful Notation : $n! = n(n-1)(n-2)\dots\dots 3.\,2.\,1$; $n! = n.(n-1)!$ $0! = 1! = 1$; $(2n)! = 2^n.\, n!\,[1.\,3.\,5.\,7\dots(2n-1)]$ Note that factorials of negative integers are not defined.

(ii) If ${}^nP_r$ denotes the number of permutations of n different things, taking r at a time, then three people then the number of ways $= \frac{(3n)!}{(n!)^3}$.

(vi) The number of permutations of n things taken all at a time when p of them are similar & of one type, q of them are similar & of another type, r of them are similar & of a third type & the remaining $n - (p + q + r)$ are all different is : $\frac{n!}{p!q!r!}$.www.MathsBySuhag.com , www.TekoClasses.com

(vii) The number of circular permutations of n different things taken all at a time is ; $(n - 1)!$. If clockwise & anti–clockwise circular permutations are considered to be same, then it is $\frac{(n-1)!}{2}$.

Note : Number of circular permutations of n things when p alike and the rest different taken all at a time distinguishing clockwise and anticlockwise arrangement is $\frac{(n-1)!}{p!}$.

(viii) Given n different objects, the number of ways of selecting atleast one of them is , ${}^nC_1 + {}^nC_2 + {}^nC_3 + \ldots + {}^nC_n = 2^n - 1$. This can also be stated as the total number of combinations of n distinct things.

(ix) Total number of ways in which it is possible to make a selection by taking some or all out of $p + q + r + \ldots$ things , where p are alike of one kind, q alike of a second kind , r alike of third kind & so on is given by : $(p + 1)(q + 1)(r + 1)\ldots\ldots - 1$.www.MathsBySuhag.com , www.TekoClasses.com

(x) Number of ways in which it is possible to make a selection of $m + n + p = N$ things , where p are alike of one kind , m alike of second kind & n alike of third kind taken r at a time is given by coefficient of x^r in the expansion of $(1 + x + x^2 + \ldots + x^p)(1 + x + x^2 + \ldots + x^m)(1 + x + x^2 + \ldots + x^n)$.

Note : Remember that coefficient of x^r in $(1 - x)^{-n} = {}^{n+r-1}C_r$ $(n \in N)$. For example the number of ways in which a selection of four letters can be made from the letters of the word **PROPORTION** is given by coefficient of x^4 in $(1 + x + x^2 + x^3)(1 + x + x^2)(1 + x + x^2)(1 + x)(1 + x)(1 + x)$.

(xi) Number of ways in which n distinct things can be distributed to p persons if there is no restriction to the number of things received by men $= p^n$.www.MathsBySuhag.com , www.TekoClasses.com

(xii) Number of ways in which n identical things may be distributed among p persons if each person may receive none , one or more things is ; ${}^{n+p-1}C_n$.

(xiii) **a.** ${}^nC_r = {}^nC_{n-r}$; ${}^nC_0 = {}^nC_n = 1$; **b.** ${}^nC_x = {}^nC_y \Rightarrow x = y$ or $x + y = n$ **c.** ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

(xiv) ${}^nC_r$ is maximum if : (a) $r = \frac{n}{2}$ if n is even. (b) $r = \frac{n-1}{2}$ or $\frac{n+1}{2}$ if n is odd.

(xv) Let $N = p^{a} \cdot q^{b} \cdot r^{c} \ldots$ where p , q , r...... are distinct primes & a , b , c..... are natural numbers then:

(a) The total numbers of divisors of N including 1 & N is $= (a + 1)(b + 1)(c + 1)\ldots$

(b) The sum of these divisors is

- Committee
- Geometrical problems
- problems on digits
- Problems on letters from a word

5.DETERMINANT

1 The symbol $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ is called the determinant of order two .

Its value is given by : $D = a_1 b_2 - a_2 b_1$

2. The symbol $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ is called the determinant of order three .

Its value can be found as : $D = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$ **OR**

$D = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$ .... and so on .In this manner we can expand a determinant in 6 ways using elements of ; R_1 , R_2 , R_3 or C_1 , C_2 , C_3 .

3. Following examples of short hand writing large expressions are :

(i) The lines : $a_1x + b_1y + c_1 = 0$........ (1)
$a_2x + b_2y + c_2 = 0$........ (2)
$a_3x + b_3y + c_3 = 0$........ (3)

are concurrent if , $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$.

Condition for the consistency of three simultaneous linear equations in 2 variables.

(ii) $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of straight lines if

$abc + 2fgh - af^2 - bg^2 - ch^2 = 0 = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$

(iii) Area of a triangle whose vertices are (x_r , y_r) ; $r = 1 , 2 , 3$ is :

$D = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$ If $D = 0$ then the three points are collinear .

is **Skew symmetric** determinant but $D' = D \Rightarrow 2D = 0 \Rightarrow D = 0 \Rightarrow$ Skew symmetric determinant of third order has the value zero .www.MathsBySuhag.com , www.TekoClasses.com

P–2 : If any two rows (or columns) of a determinant be interchanged , the value of determinant is changed in sign only . e.g.

$$\text{Let } D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \;\&\; D' = \begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad \text{Then } D' = -D .$$

P–3 : If a determinant has any two rows (or columns) identical , then its value is zero . e.g.

$$\text{Let } D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad \text{then it can be verified that } D = 0.$$

P–4 : If all the elements of any row (or column) be multiplied by the same number , then the determinant is multiplied by that number.

$$\text{e.g. If } D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \text{ and } D' = \begin{vmatrix} Ka_1 & Kb_1 & Kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad \text{Then } D' = KD$$

P–5 : If each element of any row (or column) can be expressed as a sum of two terms then the determinant can be expressed as the sum of two determinants . e.g.

$$\begin{vmatrix} a_1+x & b_1+y & c_1+z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} x & y & z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

P– 6 : The value of a determinant is not altered by adding to the elements of any row (or column) the same multiples of the corresponding elements of any other row (or column) .e.g. Let D

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \text{ and } D' = \begin{vmatrix} a_1+ma_2 & b_1+mb_2 & c_1+mc_2 \\ a_2 & b_2 & c_2 \\ a_3+na_1 & b_3+nb_1 & c_3+nc_1 \end{vmatrix} . \text{ Then } D' = D .$$

Note : that while applying this property **ATLEAST ONE ROW (OR COLUMN)** must remain unchanged .**P– 7 :** If by putting $x = a$ the value of a determinant vanishes then $(x - a)$ is a factor of the determinant .

7.Multiplication Of Two Determinants :(i) $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \times \begin{vmatrix} l_1 & m_1 \\ l_2 & m_2 \end{vmatrix} = \begin{vmatrix} a_1 l_1 + b_1 l_2 & a_1 m_1 + b_1 m_2 \\ a_2 l_1 + b_2 l_2 & a_2 m_1 + b_2 m_2 \end{vmatrix}$

$\frac{a_1}{a_2} = \frac{b_1}{b_2} \ne \frac{c_1}{c_2} \Rightarrow$Given equations are inconsistent& $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} \Rightarrow$Given equations are dependent

9. **Cramer's Rule :[Simultaneous Equations Involving Three Unknowns]**

Let ,$a_1x + b_1y + c_1z = d_1$ ...(I) ; $a_2x + b_2y + c_2z = d_2$ ...(II) ; $a_3x + b_3y + c_3z = d_3$ ...(III)

Then , $x = \frac{D_1}{D}$, $Y = \frac{D_2}{D}$, $Z = \frac{D_3}{D}$.

$$\text{Where } D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} ; D_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} ; D_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} \;\&\; D_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

Note :(a) If $D \ne 0$ and alteast one of $D_1 , D_2 , D_3 \ne 0$, then the given system of equations are consistent and have unique non trivial solution .

(b) If $D \ne 0$ & $D_1 = D_2 = D_3 = 0$, then the given system of equations are consistent and have trivial solution only .www.MathsBySuhag.com , www.TekoClasses.com

(c) If $D = D_1 = D_2 = D_3 = 0$, then the given system of equations are consistentand have infinite solutions . In case $\left.\begin{matrix} a_1x+b_1y+c_1z=d_1 \\ a_2x+b_2y+c_2z=d_2 \\ a_3x+b_3y+c_3z=d_3 \end{matrix}\right\}$ represents these parallel planes then also $D = D_1 = D_2 = D_3 = 0$ but the system is inconsistent.

(d) If $D = 0$ but atleast one of D_1 , D_2 , D_3 is not zero then the equations are inconsistent and have no solution .

10. If x , y , z are not all zero , the condition for $a_1x + b_1y + c_1z = 0$; $a_2x + b_2y + c_2z = 0$ & $a_3x + b_3y + c_3z = 0$ to be consistent in x , y , z is that $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$.Remember that if a given system of linear equations have **Only Zero** Solution for all its variables then the given equations are said to have **TRIVIAL SOLUTION**

6.MATRICES

USEFUL IN STUDY OF SCIENCE, ECONOMICS AND ENGINEERING

1. **Definition :** Rectangular array of mn numbers . Unlike determinants it has no value.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & \cdots\cdots & a_{2n} \\ : & : & : & : \\ a_{m1} & a_{m2} & \cdots\cdots & a_{mn} \end{bmatrix} \quad \text{or} \quad \begin{pmatrix} a_{11} & a_{12} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & \cdots\cdots & a_{2n} \\ : & : & : & : \\ a_{m1} & a_{m2} & \cdots\cdots & a_{mn} \end{pmatrix}$$

(e) **Verical Matrix :** A matrix of order $m \times n$ is a vertical matrix if $m > n$. $\begin{bmatrix} 1 & 1 \\ 3 & 6 \\ 2 & 4 \end{bmatrix}$

(f) **Square Matrix : (Order n)** If number of row = number of column $\Rightarrow$ a square matrix.

Note

(i) In a square matrix the pair of elements a_{ij} & a_{ji} are called **Conjugate Elements** .e.g. $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

(ii) The elements $a_{11}, a_{22}, a_{33}, \ldots\ldots a_{nn}$ are called **Diagonal Elements** . The line along which the diagonal elements lie is called **" Principal or Leading "** diagonal. The qty $\Sigma\, a_{ii}$ = trace of the matrice written as , i.e. $t_r A$ Triangular Matrix Diagonal Matrix denote as $d_{dia}(d_1, d_2, \ldots.., d_n)$ all elements except the leading diagonal are zero diagonal Matrix Unit or Identity Matrix

Note: Min. number of zeros in a diagonal matrix of order n = n(n – 1) "It is to be noted that with square matrix there is a corresponding determinant formed by the elements of A in the same order."

3. Equality Of Matrices : Let $A = [a_{ij}]$ & $B = [b_{ij}]$ are equal if ,
(i) both have the same order . **(ii)** $a_{ij} = b_{ij}$ for each pair of i & j.

4. Algebra Of Matrices : Addition : $A + B = \left[a_{ij} + b_{ij}\right]$ where A & B are of the same type. (order)

(a) Addition of matrices is commutative. i.e. $A + B = B + A$, $A = m \times n$; $B = m \times n$

(b) Matrix addition is associative . $(A + B) + C = A + (B + C)$ **Note :** A , B & C are of the same type.

(c) Additive inverse. If $A + B = \mathbf{O} = B + A$ $A = m \times n$

5. Multiplication Of A Matrix By A Scalar : If $A = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$ $kA = \begin{bmatrix} ka & kb & kc \\ kb & kc & ka \\ kc & ka & kb \end{bmatrix}$

6. Multiplication Of Matrices : (Row by Column) AB exists if, $A = m \times n$ & $B = n \times p$ 2×3 3×3
AB exists, but BA does not $\Rightarrow AB \neq BA$

Note : In the product AB , $\begin{cases} A = \text{prefactor} \\ B = \text{postfactor} \end{cases}$ $A = (a_1, a_2, \ldots\ldots a_n)$ & $B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$ $1 \times n$ $n \times 1$ A

$B = [a_1 b_1 + a_2 b_2 + \ldots\ldots + a_n b_n]$ If $A = \left[a_{ij}\right]$ $m \times n$ & $B = \left[b_{ij}\right]$ $n \times p$

matrix , then $(AB)_{ij} = \sum_{r=1}^{n} a_{ir} \cdot b_{rj}$ **<u>Properties Of Matrix Multiplication :</u>**

6. MATRIX POLYNOMIAL :
If $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots\ldots + a_nx^0$ then we define a matrix polynomial $f(A) = a_0A^n + a_1A^{n-1} + a_2A^{n-2} + \ldots.. + a_nI^n$ where A is the given square matrix. If f (A) is the null matrix then A is called the zero or root of the polynomial f (x).

DEFINITIONS : (a) **Idempotent Matrix :** A square matrix is idempotent provided $A^2 = A$.
Note that $A^n = A$ $\forall$ $n \geq 2$, $n \in$ N.

(b) **Nilpotent Matrix:** A square matrix is said to be nilpotent matrix of order m, $m \in$ N, if $A^m = \mathbf{O}$, $A^{m-1} \neq \mathbf{O}$.

(c) **Periodic Matrix :** A square matrix is which satisfies the relation $A^{K+1} = A$, for some positive integer K, is a periodic matrix. The period of the matrix is the least value of K for which this holds true.
Note that period of an idempotent matrix is 1.

(d) **Involutary Matrix :** If $A^2 = I$, the matrix is said to be an involutary matrix.
Note that $A = A^{-1}$ for an involutary matrix.

7. The Transpose Of A Matrix : (Changing rows & columns)
Let A be any matrix . Then, $A = a_{ij}$ of order $m \times n$
$\Rightarrow$ A^T or $A' = [a_{ji}]$ for $1 \leq i \leq n$ & $1 \leq j \leq m$ of order $n \times m$
Properties of Transpose : If A^T & B^T denote the transpose of A and B ,

(a) $(A \pm B)^T = A^T \pm B^T$; note that A & B have the same order.
IMP. (b) $(AB)^T = B^T A^T$ A & B are conformable for matrix product AB.
(c) $(A^T)^T = A$ **(d)** $(kA)^T = kA^T$ k is a scalar .

General : $(A_1, A_2, \ldots\ldots A_n)^T = A_n^T, \ldots\ldots., A_2^T, A_1^T$ (reversal law for transpose)

8. Symmetric & Skew Symmetric Matrix :
A square matrix $A = \left[a_{ij}\right]$ is said to be , symmetric if , $a_{ij} = a_{ji}$ $\forall$ i & j (conjugate elements are equal) **(Note $A = A^T$)**

Note: Max. number of distinct entries in a symmetric matrix of order n is $\frac{n(n+1)}{2}$.

and skew symmetric if , $a_{ij} = -a_{ji}$ $\forall$ i & j (the pair of conjugate elements are additive inverse of each other) **(Note $A = -A^T$)** Hence If A is skew symmetric, then $a_{ii} = -a_{ii}$ $\Rightarrow$ $a_{ii} = 0$ $\forall$ i Thus the digaonal elements of a skew symmetric matrix are all zero , but not the converse .

Properties Of Symmetric & Skew Matrix : P – 1 A is symmetric if $A^T = A$ A is skew symmetric if $A^T = -A$

P – 2 $A + A^T$ is a symmetric matrix $A - A^T$ is a skew symmetric matrix . Consider $(A + A^T)^T = A^T + (A^T)^T = A^T + A = A + A^T$ $A + A^T$ is symmetric . Similarly we can prove that $A - A^T$

$(\text{adj } A) = \begin{pmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{pmatrix}$ ***V. Imp.*** Theorem : $A\,(\text{adj. } A) = (\text{adj. } A).A = |A|\, I_n$, If A be a square matrix of order n. **Note :** If A and B are non singular square matrices of same order, then

(i) $|\text{adj } A| = |A|^{n-1}$ (ii) $\text{adj}(AB) = (\text{adj } B)(\text{adj } A)$

(iii) $\text{adj}(KA) = K^{n-1}(\text{adj } A)$, K is a scalar

Inverse Of A Matrix (Reciprocal Matrix) : A square matrix A said to be invertible (non singular) if there exists a matrix B such that, $AB = I = BA$

B is called the inverse (reciprocal) of A and is denoted by A^{-1} . Thus

$A^{-1} = B \Leftrightarrow AB = I = BA$. We have , $A.(\text{adj } A) = |A|\, I_n$

$A^{-1} A\,(\text{adj } A) = A^{-1} I_n\, |A|$; $I_n\,(\text{adj } A) = A^{-1}\, |A|\, I_n$ $\therefore$ $A^{-1} = \dfrac{(\text{adj } A)}{|A|}$

Note : The necessary and sufficient condition for a square matrix A to be invertible is that $|A| \neq 0$.

Imp. Theorem : If A & B are invertible matrices ofthe same order , then $(AB)^{-1} = B^{-1} A^{-1}$. This is reversal law for inverse**Note :(i)**If A be an invertible matrix , then A^T is also invertible & $(A^T)^{-1} = (A^{-1})^T$.

(ii) If A is invertible, **(a)** $(A^{-1})^{-1} = A$; **(b)** $(A^k)^{-1} = (A^{-1})^k = A^{-k}$, $k \in N$

(iii) If A is an Orthogonal Matrix. $AA^T = I = A^TA$www.MathsBySuhag.com , www.TekoClasses.com

(iv) A square matrix is said to be **orthogonal** if , $A^{-1} = A^T$.

(v) $|A^{-1}| = \dfrac{1}{|A|}$ **SYSTEM OF EQUATION & CRITERIAN FOR CONSISTENCY**

GAUSS - JORDAN METHOD $x + y + z = 6$, $x - y + z = 2$, $2x + y - z = 1$

or $\begin{pmatrix} x+y+z \\ x-y+z \\ 2x+y-z \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ 1 \end{pmatrix}$ $\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ 1 \end{pmatrix}$

$$AX = B \Rightarrow A^{-1}AX = A^{-1}B \Rightarrow X = A^{-1}B = \frac{(\text{adj. } A).B}{|A|}.$$

Note :(1) If $|A| \neq 0$, system is consistent having unique solution

(2) If $|A| \neq 0$ & $(\text{adj } A).B \neq \mathbf{O}$ (Null matrix) , system is consistent having unique non – trivial solution .**(3)** If $|A| \neq 0$ & $(\text{adj } A).B = \mathbf{O}$ (Null matrix) , system is consistent having trivial solution

NOTE : $\log_a 1 = 0\,(a > 0 , a \neq 1)$; $\log_a a = 1\,(a > 0 , a \neq 1)$ and $\log_{1/a} a = -1$ $(a > 0 , a \neq 1)$

2. THE PRINCIPAL PROPERTIES OF LOGARITHMS : Let M & N are arbitrary posiitive numbers , $a > 0 , a \neq 1 , b > 0 , b \neq 1$ and α is any real number then ;

(i) $\log_a (M . N) = \log_a M + \log_a N$ **(ii)** $\log_a (M/N) = \log_a M - \log_a N$

(iii) $\log_a M^\alpha = \alpha . \log_a M$ **(iv)** $\log_b M = \dfrac{\log_a M}{\log_a b}$

NOTE : ✎ $\log_b a . \log_a b = 1 \Leftrightarrow \log_b a = 1/\log_a b$. ✎ $\log_b a . \log_c b . \log_a c = 1$

✎ $\log_y x . \log_z y . \log_a z = \log_a x$. ✎ $e^{\ln a^x} = a^x$

3. PROPERTIES OF MONOTONOCITY OF LOGARITHM :

(i) For $a > 1$ the inequality $0 < x < y$ & $\log_a x < \log_a y$ are equivalent.

(ii) For $0 < a < 1$ the inequality $0 < x < y$ & $\log_a x > \log_a y$ are equivalent.

(iii) If $a > 1$ then $\log_a x < p \Rightarrow 0 < x < a^p$

(iv) If $a > 1$ then $\log_a x > p \Rightarrow x > a^p$ **(v)** If $0 < a < 1$ then $\log_a x < p \Rightarrow x > a^p$

(vi) If $0 < a < 1$ then $\log_a x > p \Rightarrow 0 < x < a^p$

NOTE THAT :

✎ If the number & the base are on one side of the unity , then the logarithm is positive ; If the number & the base are on different sides of unity, then the logarithm is negative.

✎ The base of the logarithm 'a' must not equal unity otherwise numbers not equal to unity will not have a logarithm & any number will be the logarithm of unity.

✎ For a non negative number 'a' & $n \geq 2$, $n \in N$ $\sqrt[n]{a} = a^{1/n}$.

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8.PROBABILITY

THINGS TO REMEMBER :RESULT – 1

(i) ***SAMPLE–SPACE* :** The set of all possible outcomes of an experiment is called the **SAMPLE–SPACE(S)**.

(ii) ***EVENT* :** A sub set of sample–space is called an **EVENT**.

(iii) ***COMPLEMENT OF AN EVENT A* :** The set of all out comes which are in S but not in A is called the **COMPLEMENT OF THE EVENT A DENOTED BY $\overline{A}$ OR A^c**.

(iv) ***COMPOUND EVENT* :** If A & B are two given events then $A \cap B$ is called **COMPOUND EVENT** and is denoted by $A \cap B$ or AB or A & B .

(v) ***MUTUALLY EXCLUSIVE EVENTS* :** Two events are said to be **MUTUALLY EXCLUSIVE** (or disjoint or incompatible) if the occurence of one precludes (rules out) the simultaneous occurence of the other . If A & B are two mutually exclusive events then $P(A \& B) = 0$.

3R and 4G balls	E_3 : getting a G ball			
3. Throwing a pair of dice	A : throwing a doublet {11, 22, 33, 44, 55, 66} B : throwing a total of 10 or more {46, 64, 55, 56, 65, 66}	Yes	No	No
4. From a well shuffled pack of cards a card is drawn	E_1 : getting a heart E_2 : getting a spade E_3 : getting a diamond E_4 : getting a club	Yes	Yes	Yes
5. From a well shuffled pack of cards a card is drawn	A = getting a heart B = getting a face card	No	No	No

RESULT – 2www.MathsBySuhag.com , www.TekoClasses.com

AUB = A+ B = A or B denotes occurence of at least A or B. For 2 events A & B : **(See fig.1)**

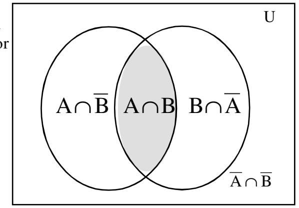


Fig . 1

(i) $P(A\cup B) = P(A) + P(B) - P(A\cap B) =$

$P(A.\overline{B}) + P(\overline{A}.B) + P(A.B) = 1 - P(\overline{A}.\overline{B})$

(ii) Opposite of " atleast A or B " is **NIETHER A NOR B** .e. $\overline{A+B}$ = 1-(A or B) = $\overline{A}\cap\overline{B}$

Note that $P(A+B) + P(\overline{A}\cap\overline{B}) = 1$.

(iii) If A & B are mutually exclusive then $P(A\cup B) = P(A) + P(B)$.

(iv) For any two events A & B, P(exactly one of A , B occurs)

$$= P\left(A\cap\overline{B}\right) + P\left(B\cap\overline{A}\right) = P(A) + P(B) - 2P(A\cap B)$$
$$= P(A\cup B) - P(A\cap B) = P\left(A^c\cup B^c\right) - P\left(A^c\cap B^c\right)$$

(v) If A & B are any two events $P(A\cap B) = P(A).P(B/A) = P(B).P(A/B)$, Where P(B/A) means conditional probability of B given A & P(A/B) means conditional probability of A given B. (This can be easily seen from the figure)

(vi) **DE MORGAN'S LAW** : – If A & B are two subsets of a universal set U , then

(a) $(A\cup B)^c = A^c\cap B^c$ & **(b)** $(A\cap B)^c = A^c\cup B^c$

(vii) $A\cup(B\cap C) = (A\cup B)\cap(A\cup C)$ & $A\cap(B\cup C) = (A\cap B)\cup(A\cap C)$

i.e $P(A\cap B) = P(B\cap C) = P(C\cap A) = 0 \Rightarrow P(A\cap B\cap C) = 0$. However the converse of this is not true.

RESULT – 4 ***INDEPENDENT EVENTS*** **:** Two events A & B are said to be independent if occurence or non occurence of one does not effect the probability of the occurence or non occurence of other.

(i) If the occurence of one event affects the probability of the occurence of the other event then the events are said to be **DEPENDENT** or **CONTINGENT** . For two independent events A and B : $P(A\cap B) = P(A).P(B)$. Often this is taken as the definition of independent events.

(ii) Three events A , B & C are independent if & only if all the following conditions hold ;

$P(A\cap B) = P(A) . P(B)$; $P(B\cap C) = P(B) . P(C)$

$P(C\cap A) = P(C) . P(A)$ & $P(A\cap B\cap C) = P(A) . P(B) . P(C)$

i.e. they must be pairwise as well as mutually independent .

Similarly for n events $A_1, A_2, A_3, \ldots\ldots A_n$ to be independent , the number of these conditions is equal to ${}^nc_2 + {}^nc_3 + \ldots.. + {}^nc_n = 2^n - n - 1$.

(iii) The probability of getting exactly r success in n independent trials is given by $P(r) = {}^nC_r\, p^r q^{n-r}$

where: p = probability of success in a single trial q = probability of failure in a single trial. note : p + q = 1

Note : Independent events are not in general mutually exclusive & vice versa.

Mutually exclusiveness can be used when the events are taken from the same experiment & independence can be used when the events are taken from different experiments .

RESULT – 5 : BAYE'S THEOREM OR TOTAL PROBABILITY THEOREM :

If an event A can occur only with one of the n mutually exclusive and exhaustive events $B_1, B_2, \ldots. B_n$ & the probabilities $P(A/B_1)$, $P(A/B_2)$ .... $P(A/B_n)$ are known then, $P(B_i/A) = \dfrac{P(B_i).P(A/B_i)}{\sum_{i=1}^{n} P(B_i).P(A/B_i)}$

PROOF : The events A occurs with one of the n mutually exclusive & exhaustive events $B_1, B_2, B_3, \ldots\ldots.. B_n$; $A = AB_1 + AB_2 + AB_3 + \ldots\ldots. + AB_n$

$P(A) = P(AB_1) + P(AB_2) + \ldots\ldots. + P(AB_n) = \sum_{i=1}^{n} P(AB_i)$

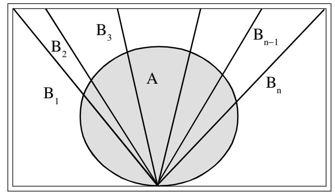


Fig . 3

NOTE : A ≡ event what we have ;

B_1 ≡ event what we want ;

$B_2, B_3, \ldots.B_n$ are alternative event .

Now, $P(AB_i) = P(A) . P(B_i/A)$

$= P(B_i) . P(A/B_i)$

$$P(B_i/A) = \frac{P(B_i) . P(A/B_i)}{P(A)} = \frac{P(B_i) . P(A/B_i)}{\sum_{i=1}^{n} P(AB_i)}$$

(ii) Mean of any probability distribution of a random variable is given by : $\mu = \dfrac{\sum p_i x_i}{\sum p_i} = \sum p_i x_i$ (Since $\Sigma p_i = 1$) **(iii)** Variance of a random variable is given by, $\sigma^2 = \sum (x_i - \mu)^2 \cdot p_i$

$\sigma^2 = \sum p_i x_i^2 - \mu^2$ (Note that $SD = +\sqrt{\sigma^2}$) **(iv)** The probability distribution for a binomial variate 'X' is given by ; $P(X = r) = {}^nC_r\, p^r q^{n-r}$ where all symbols have the same meaning as given in result 4. The recurrence formula $\dfrac{P(r+1)}{P(r)} = \dfrac{n-r}{r+1} \cdot \dfrac{p}{q}$, is very helpful for quickly computing P(1) , P(2). P(3) etc. if P(0) is known . **(v)** Mean of BPD = np ; variance of BPD = npq .

(vi) If p represents a persons chance of success in any venture and 'M' the sum of money which he will receive in case of success, then his expectations or probable value = pM expectations = pM

RESULT – 8 : GEOMETRICAL APPLICATIONS : The following statements are axiomatic :

(i) If a point is taken at random on a given staright line AB, the chance that it falls on a particular segment PQ of the line is PQ/AB . **(ii)** If a point is taken at random on the area S which includes an area σ , the chance that the point falls on σ is σ/S .

9.FUNCTIONS

THINGS TO REMEMBER : 1. GENERAL DEFINITION :

If to every value (Considered as real unless other–wise stated) of a variable x, which belongs to some collection (Set) E, there corresponds one and only one finite value of the quantity y, then y is said to be a function (Single valued) of x or a dependent variable defined on the set E ; x is the argument or independent variable .

If to every value of x belonging to some set E there corresponds one or several values of the variable y, then y is called a multiple valued function of x defined on E.Conventionally the word **"FUNCTION"** is used only as the meaning of a single valued function, if not otherwise stated. Pictorially : $\xrightarrow[\text{input}]{x} \boxed{f} \xrightarrow[\text{output}]{f(x) = y}$, y is called the image of x & x is the pre-image of y under f. Every function from $A \to B$ satisfies the following conditions .

(i) $f \subset A \times B$ **(ii)** $\forall\ a \in A \Rightarrow (a, f(a)) \in f$ and **(iii)** $(a, b) \in f$ & $(a, c) \in f \Rightarrow b = c$

2. DOMAIN, CO–DOMAIN & RANGE OF A FUNCTION :

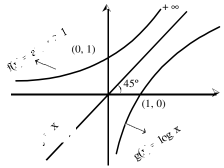
Let $f : A \to B$, then the set A is known as the domain of f & the set B is known as co-domain of f . The set of all f images of elements of A is known as the range of f . Thus

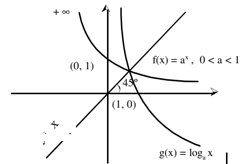
called **TRANSCEDENTAL FUNCTION** .www.MathsBySuhag.com , www.TekoClasses.com

(iii) FRACTIONAL RATIONAL FUNCTION : A rational function is a function of the form. $y = f(x) = \dfrac{g(x)}{h(x)}$, where g (x) & h (x) are polynomials & $h(x) \neq 0$.

(iv) ABSOLUTE VALUE FUNCTION : A function $y = f(x) = |x|$ is called the absolute value function or Modulus function. It is defined as : $y = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

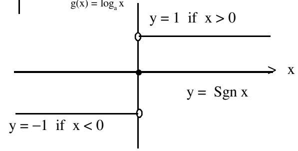
(v) EXPONENTIAL FUNCTION : A function $f(x) = a^x = e^{x \ln a}$ $(a > 0,\ a \neq 1, x \in R)$ is called anexponential function. The inverse of the exponential function is called the logarithmic function . i.e. $g(x) = \log_a x$.
Note that f(x) & g(x) are inverse of each other & their graphs are as shown .





(vi) SIGNUM FUNCTION :
A function y= f (x) = Sgn (x) is defined as follows :

$$y = f(x) = \begin{cases} 1 & \text{for } x > 0 \\ 0 & \text{for } x = 0 \\ -1 & \text{for } x < 0 \end{cases}$$

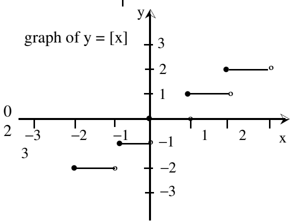


It is also written as Sgn x = |x|/ x ; $x \neq 0$; f (0) = 0

(vii) GREATEST INTEGER OR STEP UP FUNCTION :
The function y = f (x) = [x] is called the greatest integer function where [x] denotes the greatest integer less than or equal to x . Note that for :

$-1 \leq x < 0$	;	$[x] = -1$	$0 \leq x < 1$	;	$[x] = 0$
$1 \leq x < 2$	;	$[x] = 1$	$2 \leq x < 3$	;	$[x] = 2$

and so on .



Properties of greatest integer function :

(a) $[x] \leq x < [x] + 1$ and
$x - 1 < [x] \leq x$, $0 \leq x - [x] < 1$

			R^+ , if n is even
(iii)	$x^{1/n}$, $(n \in N)$	R , if n is odd $R^+ \cup \{0\}$, if n is even	R , if n is odd $R^+ \cup \{0\}$, if n is even
	Function $(y = f(x))$	**Domain** (i.e. values taken by x)	**Range** (i.e. values taken by f (x))
(iv)	$\frac{1}{x^{1/n}}$, $(n \in N)$	$R - \{0\}$, if n is odd R^+ , if n is even	$R - \{0\}$, if n is odd R^+ , if n is even

B. Trigonometric Functions

(i)	$\sin x$	R	$[-1, +1]$
(ii)	$\cos x$	R	$[-1, +1]$
(iii)	$\tan x$	$R - (2k+1)\frac{\pi}{2}, k \in I$	R
(iv)	$\sec x$	$R - (2k+1)\frac{\pi}{2}, k \in I$	$(-\infty, -1] \cup [1, \infty)$
(v)	$\text{cosec } x$	$R - k\pi, k \in I$	$(-\infty, -1] \cup [1, \infty)$
(vi)	$\cot x$	$R - k\pi, k \in I$	R

C. Inverse Circular Functions (Refer after Inverse is taught)

(i)	$\sin^{-1} x$	$[-1, +1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
(ii)	$\cos^{-1} x$	$[-1, +1]$	$[0, \pi]$
(iii)	$\tan^{-1} x$	R	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
(iv)	$\text{cosec}^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$
(v)	$\sec^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$[0, \pi] - \left\{\frac{\pi}{2}\right\}$
(vi)	$\cot^{-1} x$	R	$(0, \pi)$

D. Exponential Functions

(i)	e^x	R	R^+

	Function $(y = f(x))$	**Domain** (i.e. values taken by x)	**Range** (i.e. values taken by f (x))
(i)	$\lvert x \rvert$	R	$R^+ \cup \{0\}$
(ii)	$\frac{1}{\lvert x \rvert}$	$R - \{0\}$	R^+

I. Signum Function

$\text{sgn}(x) = \frac{\lvert x \rvert}{x}, x \neq 0$ $= 0, x = 0$	R	$\{-1, 0, 1\}$

J. Constant Function

say $f(x) = c$	R	$\{c\}$

5. EQUAL OR IDENTICAL FUNCTION :

Two functions f & g are said to be equal if :

(i) The domain of f = the domain of g.

(ii) The range of f = the range of g and

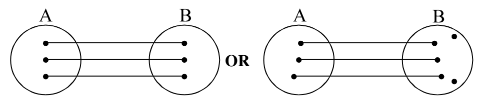
(iii) $f(x) = g(x)$, for every x belonging to their common domain. eg.

$f(x) = \frac{1}{x}$ & $g(x) = \frac{x}{x^2}$ are identical functions .

6. CLASSIFICATION OF FUNCTIONS : One−One Function (Injective mapping) :

A function $f : A \rightarrow B$ is said to be a one−one function or injective mapping if different elements of A have different f images in B . Thus for $x_1, x_2 \in A$ & $f(x_1)$, $f(x_2) \in B$, $f(x_1) = f(x_2) \Leftrightarrow x_1 = x_2$ or $x_1 \neq x_2 \Leftrightarrow f(x_1) \neq f(x_2)$.

Diagramatically an injective mapping can be shown as

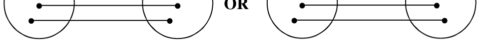


Note : (i) Any function which is entirely increasing or decreasing in whole domain, then f(x) is one−one .

(ii) If any line parallel to x−axis cuts the graph of the function atmost at one point, then the function is one−one .

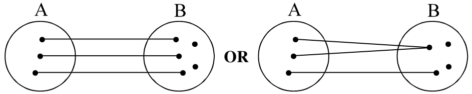
Many−one function :

A function $f : A \rightarrow B$ is said to be a many one function if two or more elements of A have the same



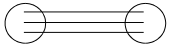
Note that : if range = co–domain, then f(x) is onto. **Into function :**
If $f : A \rightarrow B$ is such that there exists atleast one element in co–domain which is not the image of any element in domain, then f(x) is into .

Diagramatically into function can be shown as

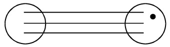


Note that : If a function is onto, it cannot be into and vice versa . A polynomial of degree even will always be into. Thus a function can be one of these four types :

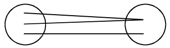
(a) one–one onto (injective & surjective)



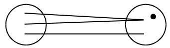
(b) one–one into (injective but not surjective)



(c) many–one onto (surjective but not injective)



(d) many–one into (neither surjective nor injective)



Note : (i) If f is both injective & surjective, then it is called a **Bijective** mapping.
The bijective functions are also named as invertible, non singular or biuniform functions.

(ii) If a set A contains n distinct elements then the number of different functions defined from $A \rightarrow A$ is n^n & out of it n ! are one one.

Identity function : The function $f : A \rightarrow A$ defined by $f(x) = x \ \forall \ x \in A$ is called the identity of A and is denoted by I_A. It is easy to observe that identity function is a bijection .

Constant function : A function $f : A \rightarrow B$ is said to be a constant function if every element of A has the same f image in B . Thus $f : A \rightarrow B$; $f(x) = c, \ \forall \ x \in A, \ c \in B$ is a constant function. Note that the range of a constant function is a singleton and a constant function may be one-one or many-one, onto or into .

7. ALGEBRAIC OPERATIONS ON FUNCTIONS :

If f & g are real valued functions of x with domain set A, B respectively, then both f & g are defined in $A \cap B$. Now we define $f + g$, $f - g$, $(f . g)$ & (f/g) as follows :

(iii) The composite of two bijections is a bijection i.e. if f & g are two bijections such that gof is defined, then gof is also a bijection.www.MathsBySuhag.com , www.TekoClasses.com

9. HOMOGENEOUS FUNCTIONS :

A function is said to be homogeneous with respect to any set of variables when each of its terms is of the same degree with respect to those variables .
For example $5x^2 + 3y^2 - xy$ is homogeneous in x & y . Symbolically if,
$f(tx , ty) = t^n . f(x, y)$ then $f(x, y)$ is homogeneous function of degree n .

10. BOUNDED FUNCTION :

A function is said to be bounded if $|f(x)| \le M$, where M is a finite quantity .

11. IMPLICIT & EXPLICIT FUNCTION :

A function defined by an equation not solved for the dependent variable is called an **Implicit Function** . For eg. the equation $x^3 + y^3 = 1$ defines y as an implicit function. If y has been expressed in terms of x alone then it is called an **Explicit Function**.

12. INVERSE OF A FUNCTION : Let $f : A \rightarrow B$ be a one–one & onto function, then their exists a unique function $g : B \rightarrow A$ such that $f(x) = y \Leftrightarrow g(y) = x, \ \forall \ x \in A \ \& \ y \in B$. Then g is said to be inverse of f . Thus $g = f^{-1} : B \rightarrow A = \{(f(x), x) \mid (x, f(x)) \in f\}$.

Properties Of Inverse Function : (i) The inverse of a bijection is unique .

(ii) If $f : A \rightarrow B$ is a bijection & $g : B \rightarrow A$ is the inverse of f, then $fog = I_B$ and $gof = I_A$, where I_A & I_B are identity functions on the sets A & B respectively.
Note that the graphs of f & g are the mirror images of each other in the line y = x . As shown in the figure given below a point (x ',y ') corresponding to $y = x^2$ $(x \ge 0)$ changes to (y ',x ') corresponding to $y = +\sqrt{x}$, the changed form of $x = \sqrt{y}$.

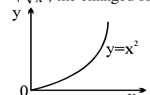


fig. 1

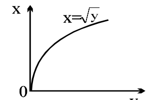


fig. 2

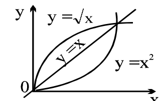


fig. 3

(iii) The inverse of a bijection is also a bijection .

(iv) If f & g two bijections $f : A \rightarrow B$, $g : B \rightarrow C$ then the inverse of gof exists and $(gof)^{-1} = f^{-1}o\, g^{-1}$

13. ODD & EVEN FUNCTIONS : If $f(-x) = f(x)$ for all x in the domain of 'f' then f is said to be an even function. e.g. $f(x) = \cos x$; $g(x) = x^2 + 3$. If $f(-x) = -f(x)$ for all x in the domain of 'f' then f is said to be an odd function. e.g. $f(x) = \sin x$; $g(x) = x^3 + x$.

Note : (a) $f(x) - f(-x) = 0 \Rightarrow f(x)$ is even & $f(x) + f(-x) = 0 \Rightarrow f(x)$ is odd .

(b) A function may neither be odd nor even .**(c)**Inverse of an even function is not defined

(d) Every even function is symmetric about the y–axis & every odd function is symmetric about the origin .

(e) If f(x) has a period p, then $\frac{1}{f(x)}$ and $\sqrt{f(x)}$ also has a period p .

(f) if f(x) has a period T then f(ax + b) has a period T/a (a > 0) .

15. GENERAL : If x, y are independent variables, then :

(i) $f(xy) = f(x) + f(y) \Rightarrow f(x) = k \ln x$ or $f(x) = 0$.

(ii) $f(xy) = f(x) . f(y) \Rightarrow f(x) = x^n$, $n \in R$ **(iii)** $f(x+y) = f(x) . f(y) \Rightarrow f(x) = a^{kx}$.

(iv) $f(x + y) = f(x) + f(y) \Rightarrow f(x) = kx$, where k is a constant .

10.INVERSE TRIGONOMETRY FUNCTION

GENERAL DEFINITION(S):1. $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$ etc. denote angles or real numbers whose sine is x , whose cosine is x and whose tangent is x, provided that the answers given are numerically smallest available . These are also written as arc sinx , arc cosx etc .

If there are two angles one positive & the other negative having same numerical value, then positive angle should be taken .

2. PRINCIPAL VALUES AND DOMAINS OF INVERSE CIRCULAR FUNCTIONS :(i) $y = \sin^{-1} x$ where $-1 \le x \le 1$; $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$ and $\sin y = x$.

(ii) $y = \cos^{-1} x$ where $-1 \le x \le 1$; $0 \le y \le \pi$ and $\cos y = x$.

(iii) $y = \tan^{-1} x$ where $x \in R$; $-\frac{\pi}{2} < x < \frac{\pi}{2}$ and $\tan y = x$.

(iv) $y = \text{cosec}^{-1} x$ where $x \le -1$ or $x \ge 1$; $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, $y \ne 0$ and $\text{cosec}\, y = x$

(v) $y = \sec^{-1} x$ where $x \le -1$ or $x \ge 1$; $0 \le y \le \pi$; $y \ne \frac{\pi}{2}$ and $\sec y = x$.

(vi) $y = \cot^{-1} x$ where $x \in R$, $0 < y < \pi$ and $\cot y = x$.

NOTE THAT : (a) 1st quadrant is common to all the inverse functions .

(b) 3rd quadrant is **not used** in inverse functions .

(c) 4th quadrant is used in the CLOCKWISE DIRECTION i.e. $-\frac{\pi}{2} \le y \le 0$.

3. PROPERTIES OF INVERSE CIRCULAR FUNCTIONS :

P–1 **(i)** $\sin(\sin^{-1} x) = x$, $-1 \le x \le 1$ **(ii)** $\cos(\cos^{-1} x) = x$, $-1 \le x \le 1$

(iii) $\tan(\tan^{-1} x) = x$, $x \in R$ **(iv)** $\sin^{-1}(\sin x) = x$, $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$

(v) $\cos^{-1}(\cos x) = x$; $0 \le x \le \pi$ **(vi)** $\tan^{-1}(\tan x) = x$; $-\frac{\pi}{2} < x < \frac{\pi}{2}$

P–6 **(i)** $\sin^{-1} x + \sin^{-1} y = \sin^{-1}\left[x\sqrt{1-y^2} + y\sqrt{1-x^2}\right]$ where $x \ge 0, y \ge 0$ & $(x^2 + y^2) \le 1$

Note that : $x^2 + y^2 \le 1 \Rightarrow 0 \le \sin^{-1} x + \sin^{-1} y \le \frac{\pi}{2}$

(ii) $\sin^{-1} x + \sin^{-1} y = \pi - \sin^{-1}\left[x\sqrt{1-y^2} + y\sqrt{1-x^2}\right]$ where $x \ge 0, y \ge 0$ & $x^2 + y^2 > 1$

Note that : $x^2 + y^2 > 1 \Rightarrow \frac{\pi}{2} < \sin^{-1} x + \sin^{-1} y < \pi$

(iii) $\sin^{-1}x - \sin^{-1}y = \sin^{-1}\left[x\sqrt{1-y^2} - y\sqrt{1-x^2}\right]$ where $x \ge 0$, $y \ge 0$

(iv) $\cos^{-1} x \pm \cos^{-1} y = \cos^{-1}\left[xy \mp \sqrt{1-x^2}\sqrt{1-y^2}\right]$ where $x \ge 0$, $y \ge 0$

P–7 If $\tan^{-1} x + \tan^{-1} y + \tan^{-1} z = \tan^{-1}\left[\frac{x+y+z-xyz}{1-xy-yz-zx}\right]$ if, $x > 0, y > 0, z > 0$ & $xy+yz+zx < 1$

Note : (i) If $\tan^{-1} x + \tan^{-1} y + \tan^{-1} z = \pi$ then $x + y + z = xyz$

(ii) If $\tan^{-1} x + \tan^{-1} y + \tan^{-1} z = \frac{\pi}{2}$ then $xy + yz + zx = 1$

P–8 $2\tan^{-1} x = \sin^{-1}\frac{2x}{1+x^2} = \cos^{-1}\frac{1-x^2}{1+x^2} = \tan^{-1}\frac{2x}{1-x^2}$ **Note very carefully that :**

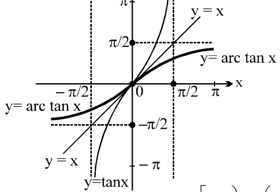
$$\sin^{-1}\frac{2x}{1+x^2} = \begin{cases} 2\tan^{-1} x & \text{if } |x| \le 1 \\ \pi - 2\tan^{-1} x & \text{if } x > 1 \\ -(\pi + 2\tan^{-1} x) & \text{if } x < -1 \end{cases} \qquad \cos^{-1}\frac{1-x^2}{1+x^2} = \begin{cases} 2\tan^{-1} x & \text{if } x \ge 0 \\ -2\tan^{-1} x & \text{if } x < 0 \end{cases}$$

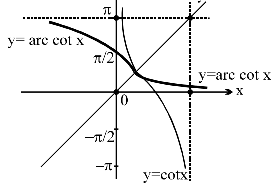
$$\tan^{-1}\frac{2x}{1-x^2} = \begin{cases} 2\tan^{-1} x & \text{if } |x| < 1 \\ \pi + 2\tan^{-1} x & \text{if } x < -1 \\ -(\pi - 2\tan^{-1} x) & \text{if } x > 1 \end{cases}$$

REMEMBER THAT : (i) $\sin^{-1} x + \sin^{-1} y + \sin^{-1} z = \frac{3\pi}{2} \Rightarrow x = y = z = 1$

(ii) $\cos^{-1} x + \cos^{-1} y + \cos^{-1} z = 3\pi \Rightarrow x = y = z = -1$

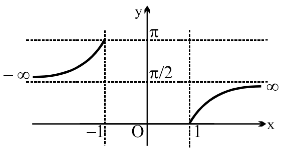
(iii) $\tan^{-1} 1 + \tan^{-1} 2 + \tan^{-1} 3 = \pi$ and $\tan^{-1} 1 + \tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{3} = \frac{\pi}{2}$

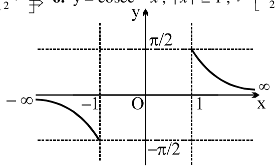




5. $y = \sec^{-1} x$, $|x| \ge 1$, $y \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$

6. $y = \text{cosec}^{-1} x$, $|x| \ge 1$, $y \in \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$



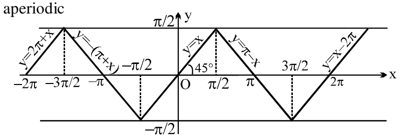


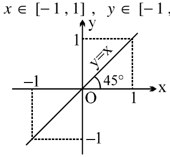
7. (a) $y = \sin^{-1}(\sin x)$, $x \in R$, $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Periodic with period 2π

7.(b) $y = \sin(\sin^{-1} x)$,

$= x$ $x \in [-1, 1]$, $y \in [-1, 1]$, y is aperiodic



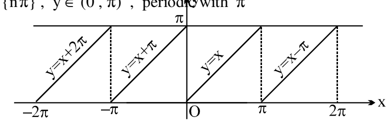


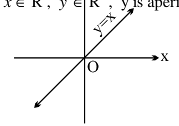
8. (a) $y = \cos^{-1}(\cos x)$, $x \in R$, $y \in [0, \pi]$,
$= x$ periodic with period 2π

8. (b) $y = \cos(\cos^{-1} x)$,
$= x$
$x \in [-1, 1]$, $y \in [-1, 1]$, y is aperiodic

$x \in R - \{n\pi\}$, $y \in (0, \pi)$, periodic with π

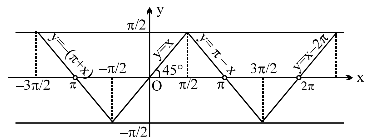
$x \in R$, $y \in R$, y is aperiodic

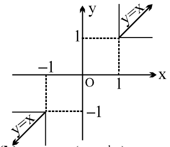




11. (a) $y = \text{cosec}^{-1}(\text{cosec}\, x)$,
$= x$
$x \in R - \{n\pi, n \in I\}$, $y \in \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$
y is periodic with period 2π

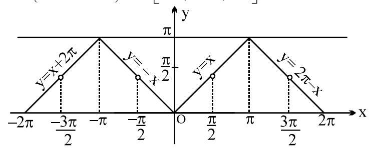
11. (b) $y = \text{cosec}(\text{cosec}^{-1} x)$,
$= x$
$|x| \ge 1$, $|y| \ge 1$, y is aperiodic

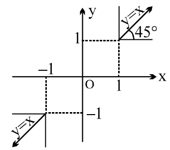




12. (a) $y = \sec^{-1}(\sec x)$,
$= x$
y is periodic with period 2π;
$x \in R - \left\{(2n-1)\frac{\pi}{2} \; n \in I\right\}$ $y \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$

12. (b) $y = \sec(\sec^{-1} x)$,
$= x$
$|x| \ge 1$; $|y| \ge 1$], y is aperiodic





3. **STANDARD LIMITS :**

(a) $\lim_{x\to 0} \frac{\sin x}{x} = 1 = \lim_{x\to 0} \frac{\tan x}{x} = \lim_{x\to 0} \frac{\tan^{-1} x}{x} = \lim_{x\to 0} \frac{\sin^{-1} x}{x}$ [Where x is measured in radians]

(b) $\lim_{x\to 0} (1+x)^{1/x} = e = \lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x$ note however the re $\lim_{\substack{h\to 0 \\ n\to\infty}} (1-h)^n = 0$ and $\lim_{\substack{h\to 0 \\ n\to\infty}} (1+h)^n \to \infty$

(c) If $\lim_{x\to a} f(x) = 1$ and $\lim_{x\to a} \phi(x) = \infty$, then ; $\lim_{x\to a} [f(x)]^{\phi(x)} = e^{\lim_{x\to a} \phi(x)[f(x)-1]}$

(d) If $\lim_{x\to a} f(x) = A > 0$ & $\lim_{x\to a} \phi(x) = B$ (a finite quantity) then ;

$\lim_{x\to a} [f(x)]^{\phi(x)} = e^z$ where $z = \lim_{x\to a} \phi(x). \ln[f(x)] = e^{B\ln A} = A^B$

(e) $\lim_{x\to 0} \frac{a^x - 1}{x} = \ln a \ (a > 0)$. In particular $\lim_{x\to 0} \frac{e^x - 1}{x} = 1$ **(f)** $\lim_{x\to a} \frac{x^n - a^n}{x - a} = n a^{n-1}$

4. **SQUEEZE PLAY THEOREM :**

If $f(x) \le g(x) \le h(x)\ \forall\ x$ & $\lim_{x\to a} f(x) = l = \lim_{x\to a} h(x)$ then $\lim_{x\to a} g(x) = l$.

5. **INDETERMINANT FORMS :**

$\frac{0}{0}, \frac{\infty}{\infty}, 0 \times \infty, 0^\circ, \infty^\circ, \infty - \infty$ and 1^∞

Note :

(i) We cannot plot ∞ on the paper. Infinity (∞) is a symbol & not a number. It does not obey the laws of elementry algebra.

(ii) $\infty + \infty = \infty$ **(iii)** $\infty \times \infty = \infty$ **(iv)** $(a/\infty) = 0$ if a is finite

(v) $\frac{a}{0}$ is not defined , if $a \ne 0$. **(vi)** $ab = 0$, if & only if $a = 0$ or $b = 0$ and a & b are finite.

6. The following strategies should be born in mind for evaluating the limits:

(a) Factorisation **(b)** Rationalisation or double rationalisation

(c) Use of trigonometric transformation ;
appropriate substitution and using standard limits

(d) Expansion of function like Binomial expansion, exponential & logarithmic expansion, expansion of sinx , cosx , tanx should be remembered by heart & are given below :

(i) $a^x = 1 + \frac{x \ln a}{1!} + \frac{x^2 \ln^2 a}{2!} + \frac{x^3 \ln^3 a}{3!} + \ldots\ldots\ldots a > 0$ **(ii)** $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots\ldots\ldots$

(iii) $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots\ldots$ for $-1 < x \le 1$ **(iv)** $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\ldots.$

(ii) f(x) is not defined at x= c

(iii) $\lim_{x\to c} f(x) \ne f(c)$

Geometrically, the graph of the function will exhibit a break at x= c.
The graph as shown is discontinuous at x = 1 , 2 and 3.

3. **Types of Discontinuities :**

Type - 1: (Removable type of discontinuities)

In case $\lim_{x\to c} f(x)$ exists but is not equal to f(c) then the function is said to have a removable discontinuity or discontinuity of the first kind. In this case we can redefine the function such that $\lim_{x\to c} f(x) = f(c)$ & make it continuous at x= c. Removable type of discontinuity can be further classified as :

(a) **MISSING POINT DISCONTINUITY :** Where $\lim_{x\to a} f(x)$ exists finitely but f(a) is not defined.

e.g. $f(x) = \frac{(1-x)(9-x^2)}{(1-x)}$ has a missing point discontinuity at x = 1 , and $f(x) = \frac{\sin x}{x}$ has a missing point discontinuity at x = 0

(b) **ISOLATED POINT DISCONTINUITY :** Where $\lim_{x\to a} f(x)$ exists & f(a) also exists but ; $\lim_{x\to a} \ne f(a)$. e.g. $f(x) = \frac{x^2-16}{x-4}$, $x \ne 4$ & f (4) = 9 has an isolated point discontinuity at x = 4.

Similarly $f(x) = [x] + [-x] = \begin{cases} 0 & \text{if } x \in I \\ -1 & \text{if } x \notin I \end{cases}$ has an isolated point discontinuity at all $x \in I$.

Type-2: (Non - Removable type of discontinuities)

In case $\lim_{x\to c} f(x)$ does not exist then it is not possible to make the function continuous by redefining it.

Such discontinuities are known as non - removable discontinuity oR
discontinuity of the 2nd kind. Non-removable type of discontinuity can be further classified as :

(a) Finite discontinuity e.g. f(x) = x – [x] at all integral x ; $f(x) = \tan^{-1}\frac{1}{x}$ at x = 0 and $f(x) = \frac{1}{1+2^{\frac{1}{x}}}$ at x = 0 (note that $f(0^+) = 0$; $f(0^-) = 1$)

domains.

6. If f & g are two functions that are continuous at x= c then the functions defined by : $F_1(x) = f(x) \pm g(x)$; $F_2(x) = K f(x)$, K any real number ; $F_3(x) = f(x).g(x)$ are also continuous at x= c.

Further, if g (c) is not zero, then $F_4(x) = \frac{f(x)}{g(x)}$ is also continuous at x= c.

7. **<u>The intermediate value theorem:</u>**

Suppose f(x) is continuous on an interval I , and a and b are any two points of I. Then if y_0 is a number between f(a) and f(b) , their exists a number c between a and b such that $f(c) = y_0$. **Note Very Carefully That :** www.MathsBySuhag.com , www.TekoClasses.com

(a) If f(x) is continuous & g(x) is discontinuous at x = a then the product function $\phi(x) = f(x) \cdot g(x)$ is not necessarily be discontinuous at x = a. e.g. f(x) = x & $g(x) = \begin{cases} \sin\frac{\pi}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$

(b) If f(x) and g(x) both are discontinuous at x = a then the product function $\phi(x) = f(x) \cdot g(x)$ is not necessarily be discontinuous at x = a. e.g. $f(x) = -g(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}$

(c) Point functions are to be treated as discontinuous. eg. $f(x) = \sqrt{1-x} + \sqrt{x-1}$ is not continuous at x = 1.

(d) A Continuous function whose domain is closed must have a range also in closed interval.

(e) If f is continuous at x = c & g is continuous at x = f(c) then the composite g[f(x)] is continuous at x = c. eg. $f(x) = \frac{x \sin x}{x^2+2}$ & $g(x) = |x|$ are continuous at x = 0 , hence the composite (gof) (x) $= \left|\frac{x \sin x}{x^2+2}\right|$ will also be continuous at x = 0 .www.MathsBySuhag.com , www.TekoClasses.com

7. **Continuity In An Interval :**

(a) A function f is said to be continuous in (a , b) if f is continuous at each & every point $\in (a,b)$.

(b) A function f is said to be continuous in a closed interval $[a, b]$ if :

(i) f is continuous in the open interval (a , b) &

(ii) f is right continuous at **'a'** i.e. $\underset{x \to a^+}{\text{Limit}}\ f(x) = f(a)$ = a finite quantity.

(iii) f is left continuous at **'b'** i.e. $\underset{x \to b^-}{\text{Limit}}\ f(x) = f(b)$ = a finite quantity.

Note that a function f which is continuous in $[a, b]$ possesses the following properties :

(i) If f(a) & f(b) possess opposite signs, then there exists at least one solution of the equation f(x) = 0 in the

$f'(a^+) = \underset{h \to 0^+}{\text{Limit}} \frac{f(a+h)-f(a)}{h}$,

provided the limit exists & is finite.www.MathsBySuhag.com , www.TekoClasses.com

(ii) The left hand derivative : of f at x = a denoted by

$f'(a^+)$ is defined by : $f'(a^-) = \underset{h \to 0^+}{\text{Limit}} \frac{f(a-h)-f(a)}{-h}$, Provided the limit exists & is finite.

We also write $f'(a^+) = f'_+(a)$ & $f'(a^-) = f'_-(a)$.

* This geomtrically means that a unique tangent with finite slope can be drawn at x = a as shown in the figure.www.MathsBySuhag.com , www.TekoClasses.com

(iii) **Derivability & Continuity :**

(a) If $f'(a)$ exists then f(x) is derivable at x= a $\Rightarrow$ f(x) is continuous at x = a.

(b) If a function f is derivable at x then f is continuous at x.

For : $f'(x) = \underset{h \to 0}{\text{Limit}} \frac{f(x+h)-f(x)}{h}$ exists.

Also $f(x+h)-f(x) = \frac{f(x+h)-f(x)}{h} . h [h \neq 0]$

Therefore : $[f(x+h)-f(x)] = \underset{h \to 0}{\text{Limit}} \frac{f(x+h)-f(x)}{h} . h = f'(x) . 0 = 0$

Therefore $\underset{h \to 0}{\text{Limit}} [f(x+h)-f(x)] = 0 \Rightarrow \underset{h \to 0}{\text{Limit}}\ f(x+h) = f(x) \Rightarrow$ f is continuous at x.

Note : If f(x) is derivable for every point of its domain of definition, then it is continuous in that domain.

The Converse of the above result is not true :

" IF f IS CONTINUOUS AT x , THEN f IS DERIVABLE AT x " IS NOT TRUE.

e.g. the functions $f(x) = |x|$ & $g(x) = x \sin\frac{1}{x}$; $x \neq 0$ & $g(0) = 0$ are continuous at x = 0 but not derivable at x = 0.

Note Carefully :

(a) Let $f'_+(a) = p$ & $f'_-(a) = q$ where p & q are finite then :

(i) $p = q \Rightarrow$ f is derivable at x = a $\Rightarrow$ f is continuous at x = a.

(ii) $p \neq q \Rightarrow$ f is not derivable at x = a.

It is very important to note that f may be still continuous at x = a.

In short, for a function f :

Differentiability $\Rightarrow$ Continuity ; Continuity $\not\Rightarrow$ derivability ;

e.g. $f(x) = \begin{cases} x^2 \sin\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

6. **A surprising result :** Suppose that the function f (x) and g (x) defined in the interval (x_1, x_2) containing the point x_0, and if f is differentiable at $x = x_0$ with $f(x_0) = 0$ together with g is continuous as $x = x_0$ then the function $F(x) = f(x) \cdot g(x)$ is differentiable at $x = x_0$
e.g. $F(x) = \sin x \cdot x^{2/3}$ is differentiable at x = 0.

12. DIFFERENTIATION & L' HOSPITAL RULE

1. **DEFINITION :**
If x and $x+h$ belong to the domain of a function f defined by $y = f(x)$, then

$\underset{h\to 0}{\text{Limit}} \frac{f(x+h)-f(x)}{h}$ if it exists, is called the **Derivative** of f at x & is denoted by $f'(x)$ or $\frac{dy}{dx}$. We have therefore, $f'(x) = \underset{h\to 0}{\text{Limit}} \frac{f(x+h)-f(x)}{h}$

2. The derivative of a given function f at a point x = a of its domain is defined as :

$\underset{h\to 0}{\text{Limit}} \frac{f(a+h)-f(a)}{h}$, provided the limit exists & is denoted by $f'(a)$.

Note that alternatively, we can define $f'(a) = \underset{x\to a}{\text{Limit}} \frac{f(x)-f(a)}{x-a}$, provided the limit exists.

3. **DERIVATIVE OF f(x) FROM THE FIRST PRINCIPLE /ab INITIO METHOD:**

If f(x) is a derivable function then, $\underset{\delta x\to 0}{\text{Limit}} \frac{\delta y}{\delta x} = \underset{\delta x\to 0}{\text{Limit}} \frac{f(x+\delta x)-f(x)}{\delta x} = f'(x) = \frac{dy}{dx}$

4. **THEOREMS ON DERIVATIVES :**
If u and v are derivable function of x, then,

(i) $\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$ **(ii)** $\frac{d}{dx}(K\,u) = K\frac{du}{dx}$, where K is any constant

(iii) $\frac{d}{dx}(u\,.\,v) = u\frac{dv}{dx} \pm v\frac{du}{dx}$ known as " **Product Rule** "

$\frac{dx}{dy} = 1 / \left(\frac{dy}{dx}\right)$ or $\frac{dy}{dx} \cdot \frac{dx}{dy} = 1$ or $\frac{dy}{dx} = 1 / \left(\frac{dx}{dy}\right)$ $[\frac{dx}{dy} \neq 0]$

(b) **Results :**

(i) $D(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$, $-1<x<1$ **(ii)** $D(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$, $-1<x<1$

(iii) $D(\tan^{-1} x) = \frac{1}{1+x^2}$, $x \in R$ **(iv)** $D(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$, $|x|>1$

(v) $D(\text{cosec}^{-1} x) = \frac{-1}{|x|\sqrt{x^2-1}}$, $|x|>1$ **(vi)** $D(\cot^{-1} x) = \frac{-1}{1+x^2}$, $x \in R$

Note : In general if $y = f(u)$ then $\frac{dy}{dx} = f'(u) \cdot \frac{du}{dx}$.

7. **LOGARITHMIC DIFFERENTIATION :** To find the derivative of :
(i) a function which is the product or quotient of a number of functions **OR**
(ii) a function of the form $[f(x)]^{g(x)}$ where f & g are both derivable, it will be found convinient to take the logarithm of the function first & then differentiate. This is called **Logarithmic Differentiation.**

8. **IMPLICIT DIFFERENTIATION :** $\phi(x, y) = 0$
(i) In order to find dy/dx, in the case of implicit functions, we differentiate each term w.r.t. x regarding y as a functions of x & then collect terms in dy/dx together on one side to finally find dy/dx.www.MathsBySuhag.com , www.TekoClasses.com
(ii) In answers of dy/dx in the case of implicit functions, both x & y are present .

9. **PARAMETRIC DIFFERENTIATION :**

If $y = f(\theta)$ & $x = g(\theta)$ where θ is a parameter, then $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$.

10. **DERIVATIVE OF A FUNCTION W.R.T. ANOTHER FUNCTION :**

Let $y = f(x)$; $z = g(x)$ then $\frac{dy}{dz} = \frac{dy/dx}{dz/dx} = \frac{f'(x)}{g'(x)}$.

11. **DERIVATIVES OF ORDER TWO & THREE :**
Let a function y = f(x) be defined on an open interval (a, b). It's derivative, if it exists on (a, b) is a certain function $f'(x)$ [or (dy/dx) or y] & is called the first derivative of y w.r.t. x.If it happens that the first derivative has a derivative on (a , b) then this derivative is called the second derivative of y w. r. t. x & is denoted by $f''(x)$ or (d^2y/dx^2) or y''.Similarly, the 3rd order

$\lim\limits_{x\to a}\frac{f(x)}{g(x)} = \lim\limits_{x\to a}\frac{f'(x)}{g'(x)} = \lim\limits_{x\to a}\frac{f''(x)}{g''(x)}$ & soon till indeterminant form vanishes.

14. ANALYSIS AND GRAPHS OF SOME USEFUL FUNCTIONS :

(i) $y = f(x) = \sin^{-1}\left(\frac{2x}{1+x^2}\right) = \begin{cases} 2\tan^{-1}x & |x| \le 1 \\ \pi - 2\tan^{-1}x & x > 1 \\ -\left(\pi + 2\tan^{-1}x\right) & x < -1 \end{cases}$

HIGHLIGHTS :

(a) Domain is $x \in R$ & range is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(b) f is continuous for all x but not diff. at $x = 1, -1$

(c) $\frac{dy}{dx} = \begin{cases} \frac{2}{1+x^2} & \text{for } |x| < 1 \\ \text{non existent} & \text{for } |x| = 1 \\ -\frac{2}{1+x^2} & \text{for } |x| > 1 \end{cases}$

(d) I in $(-1, 1)$ & D in $(-\infty, -1) \cup (1, \infty)$

ii) Consider $y = f(x) = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = \begin{cases} 2\tan^{-1}x & \text{if } x \ge 0 \\ -2\tan^{-1}x & \text{if } x < 0 \end{cases}$

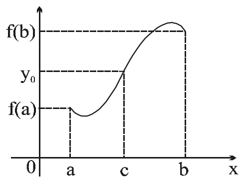
HIGHLIGHTS :

(a) Domain is $x \in R$ & range is $[0, \pi)$

(b) Continuous for all x but not diff. at $x = 0$

(c) $\frac{dy}{dx} = \begin{cases} \frac{2}{1+x^2} & \text{for } x > 0 \\ \text{non existent} & \text{for } x = 0 \\ -\frac{2}{1+x^2} & \text{for } x < 0 \end{cases}$

(d) I in $(0, \infty)$ & D in $(-\infty, 0)$



HIGHLIGHTS : www.MathsBySuhag.com , www.TekoClasses.com

(a) Domain is $x \in [-1, 1]$ & range is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(b) Not derivable at $|x| = \frac{1}{2}$

(c) $\frac{dy}{dx} = \begin{cases} \frac{3}{\sqrt{1-x^2}} & \text{if } x \in \left(-\frac{1}{2}, \frac{1}{2}\right) \\ -\frac{3}{\sqrt{1-x^2}} & \text{if } x \in \left(-1, -\frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right) \end{cases}$

(d) Continuous everywhere in its domain

(v) $y = f(x) = \cos^{-1}(4x^3 - 3x) = \begin{cases} 3\cos^{-1}x - 2\pi & \text{if } -1 \le x \le -\frac{1}{2} \\ 2\pi - 3\cos^{-1}x & \text{if } -\frac{1}{2} \le x \le \frac{1}{2} \\ 3\cos^{-1}x & \text{if } \frac{1}{2} \le x \le 1 \end{cases}$

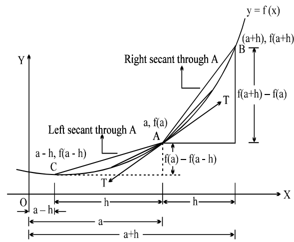
HIGHLIGHTS :

(a) Domain is $x \in [-1, 1]$ & range is $[0, \pi]$

(b) Continuous everywhere in its domain but not derivable at $x = \frac{1}{2}, -\frac{1}{2}$

(c) I in $\left(-\frac{1}{2}, \frac{1}{2}\right)$ & D in $\left(\frac{1}{2}, 1\right] \cup \left[-1, -\frac{1}{2}\right)$

(d) $\frac{dy}{dx} = \begin{cases} \frac{3}{\sqrt{1-x^2}} & \text{if } x \in \left(-\frac{1}{2}, \frac{1}{2}\right) \\ -\frac{3}{\sqrt{1-x^2}} & \text{if } x \in \left(-1, -\frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right) \end{cases}$



GENERAL NOTE :
Concavity in each case is decided by the sign of 2^{nd} derivative as :

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1.The point P (x_1, y_1) will satisfy the equation of the curve & the eqation of tangent & normal line.

2.If the tangent at any point P on the curve is // to the axis of x then $dy/dx = 0$ at the point P.

3.If the tangent at any point on the curve is parallel to the axis of y, then $dy/dx = \infty$ or $dx/dy = 0$.

4. If the tangent at any point on the curve is equally inclined to both the axes then $dy/dx = \pm 1$.

5. If the tangent at any point makes equal intercept on the coordinate axes then $dy/dx = -1$.

6. Tangent to a curve at the point P (x_1, y_1) can be drawn even through dy/dx at P does not exist. e.g. $x = 0$ is a tangent to $y = x^{2/3}$ at (0, 0).

7. If a curve passing through the origin be given by a rational integral algebraic equation, the equation of the tangent (or tangents) at the origin is obtained by equating to zero the terms of the lowest degree in the equation. e.g. If the equation of a curve be $x^2 - y^2 + x^3 + 3x^2y - y^3 = 0$, the tangents at the origin are given by $x^2 - y^2 = 0$ i.e. $x + y = 0$ and $x - y = 0$.

IV Angle of intersection between two curves is defined as the angle between the 2 tangents drawn to the 2 curves at their point of intersection. If the angle between two curves is 90° every where then they are called **ORTHOGONAL** curves.

V **(a)** Length of the tangent (PT) = $\dfrac{y_1\sqrt{1+[f'(x_1)]^2}}{f'(x_1)}$

(b) Length of Subtangent (MT) = $\dfrac{y_1}{f'(x_1)}$

(c) Length of Normal (PN) = $y_1\sqrt{1+[f'(x_1)]^2}$

(d) Length of Subnormal (MN) = $y_1 f'(x_1)$

VI **DIFFERENTIALS :**

The differential of a function is equal to its derivative multiplied by the differential of the independent variable. Thus if, $y = \tan x$ then $dy = \sec^2 x\, dx$.

In general $dy = f'(x)\, dx$.

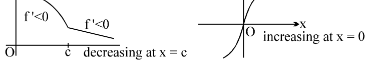
Note that : $d(c) = 0$ where 'c' is a constant.

$d(u + v - w) = du + dv - dw$ $d(uv) = u\,dv + v\,du$

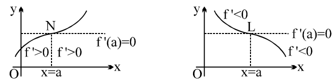
Note :

1. For the independent variable 'x', increment Δx and differential dx are equal but this is not the case with the dependent variable 'y' i.e. $\Delta y \neq dy$.

2. The relation $dy = f'(x)\, dx$ can be written as $\dfrac{dy}{dx} = f'(x)$; thus the quotient of the differentials of 'y' and



Note : If $f'(a) = 0$, then for $x = a$ the function may be still increasing or it may be decreasing as shown. It has to be identified by a seperate rule. e.g. $f(x) = x^3$ is increasing at every point. Note that, $\mathbf{dy/dx = 3x^2}$.



5. **Tests for Increasing & Decreasing of a function in an interval :**

SUFFICIENCY TEST : If the derivative function $f'(x)$ in an interval (a , b) is every where positive, then the function f (x) in this interval is Increasing ;www.MathsBySuhag.com , www.TekoClasses.com

If $f'(x)$ is every where negative, then f (x) is Decreasing.

General Note :

(1) If a function is invertible it has to be either increasing or decreasing.

(2) If a function is continuous the intervals in which it rises and falls may be separated by points at which its derivative fails to exist.

(3) If f is increasing in [a, b] and is continuous then f (b) is the greatest and f (c) is the least value of f in [a, b]. Similarly if f is decreasing in [a, b] then f (a) is the greatest value and f (b) is the least value.

6. **(a)** **ROLLE'S THEOREM :**

Let f(x) be a function of x subject to the following conditions :

(i) f(x) is a continuous function of x in the closed interval of $a \le x \le b$.

(ii) $f'(x)$ exists for every point in the open interval $a < x < b$.

(iii) $f(a) = f(b)$. Then there exists at least one point $x = c$ such that $a<c<b$ where $f'(c) = 0$.

Note that if f is not continuous in closed [a, b] then it may lead to the adjacent graph where all the 3 conditions of Rolles will be valid but the assertion will not be true in (a, b).

(b) **LMVT THEOREM :**

Let f(x) be a function of x subject to the following conditions :

(i) f(x) is a continuous function of x in the closed interval of $a \le x \le b$.

(ii) $f'(x)$ exists for every point in the open interval $a < x < b$.

(iii) $f(a) \neq f(b)$.

(ii) f(a) & f(b) have opposite signs.
(iii) $f'(x)$ is different from zero for all values of x between a & b.
Then there is one & only one real root of the equation $f(x) = 0$ between a & b.

MAXIMA - MINIMA

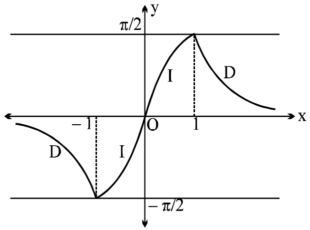
FUNCTIONS OF A SINGLE VARIABLE

HOW MAXIMA & MINIMA ARE CLASSIFIED

1. A function f(x) is said to have a maximum at x = a if f(a) is greater than every other value assumed by f(x) in the immediate neighbourhood of x = a. Symbolically

$$\left.\begin{matrix} f(a) > f(a+h) \\ f(a) > f(a-h) \end{matrix}\right] \Rightarrow x = a \text{ gives maxima for a}$$

sufficiently small positive h.



Similarly, a function f(x) is said to have a minimum value at x = b if f(b) is least than every other value assumed by f(x) in the immediate neighbourhood at x = b. Symbolically if $\left.\begin{matrix} f(b) < f(b+h) \\ f(b) < f(b-h) \end{matrix}\right] \Rightarrow x = b$ gives minima for a sufficiently small positive h.

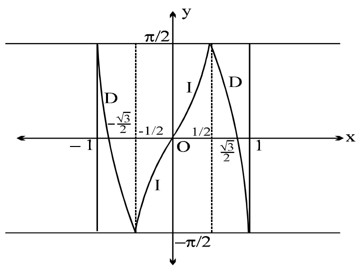
Note that :

(i) the maximum & minimum values of a function are also known as local/relative maxima or local/relative minima as these are the greatest & least values of the function relative to some neighbourhood of the point in question.

(ii) the term 'extremum' or (extremal) or 'turning value' is used both for maximum or a minimum value.

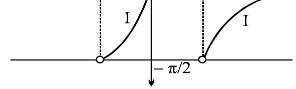
(iii) a maximum (minimum) value of a function may not be the greatest (least) value in a finite interval.

(iv) a function can have several maximum & minimum values & a minimum value may even be greater than a maximum value.



(v) maximum & minimum values of a continuous

which $f'(x) = 0$.



(iv) Critical points are those where $\frac{dy}{dx} = 0$, if it exists , or it fails to exist either by virtue of a vertical tangent or by virtue of a geometrical sharp corner but not because of discontinuity of function.

3. **SUFFICIENT CONDITION FOR EXTREME VALUES :**

$$\left.\begin{matrix} f'(c-h) > 0 \\ f'(c+h) < 0 \end{matrix}\right] \Rightarrow x = c \text{ is a point of local maxima, where } f'(c) = 0.$$

$$\text{Similarly } \left.\begin{matrix} f'(c-h) < 0 \\ f'(c+h) > 0 \end{matrix}\right] \Rightarrow x = c \text{ is a point of local minima, where } f'(c) = 0.$$

Note : If $f'(x)$ does not change sign i.e. has the same sign in a certain complete neighbourhood of c, then f(x) is either strictly increasing or decreasing throughout this neighbourhood implying that f(c) is not an extreme value of f.

4. **USE OF SECOND ORDER DERIVATIVE IN ASCERTAINING THE MAXIMA OR MINIMA:**

(a) f(c) is a minimum value of the function f, if $f'(c) = 0$ & $f''(c) > 0$.

(b) f(c) is a maximum value of the function f, $f'(c) = 0$ & $f''(c) < 0$.

Note : if $f''(c) = 0$ then the test fails. Revert back to the first order derivative check for ascertaning the maxima or minima.

5. **SUMMARY−WORKING RULE :**

FIRST :
When possible , draw a figure to illustrate the problem & label those parts that are important in the problem. Constants & variables should be clearly distinguished.

SECOND :
Write an equation for the quantity that is to be maximised or minimised. If this quantity is denoted by 'y', it must be expressed in terms of a single independent variable x. his may require some algebraic manipulations.

THIRD :
If y = f (x) is a quantity to be maximum or minimum, find those values of x for which $dy/dx = f'(x) = 0$.

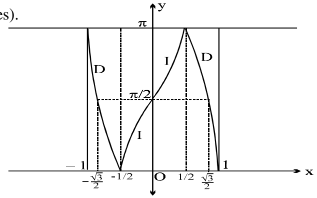
FOURTH :
Test each values of x for which $f'(x) = 0$ to determine whether it provides a maximum or minimum or neither. The usual tests are :

(a) If d^2y/dx^2 is positive when $dy/dx = 0 \Rightarrow$ y is minimum.
If d^2y/dx^2 is negative when $dy/dx = 0 \Rightarrow$ y is maximum.
If $d^2y/dx^2 = 0$ when $dy/dx = 0$, the test fails.

$y)^2 = (x-y)^2 + 4xy$

6. USEFUL FORMULAE OF MENSURATION TO REMEMBER :

- Volume of a cuboid = lbh.
- Surface area of a cuboid = $2(lb + bh + hl)$.
- Volume of a prism = area of the base x height.
- Lateral surface of a prism = perimeter of the base x height.
- Total surface of a prism = lateral surface + 2 area of the base
  (Note that lateral surfaces of a prism are all rectangles).
- Volume of a pyramid = $\frac{1}{3}$ area of the base x height.
- Curved surface of a pyramid = $\frac{1}{2}$ (perimeter of the base) x slant height.
  (Note that slant surfaces of a pyramid are triangles).
- Volume of a cone = $\frac{1}{3}\pi r^2 h$.
- Curved surface of a cylinder = $2\pi rh$.
- Total surface of a cylinder = $2\pi rh + 2\pi r^2$.
- Volume of a sphere = $\frac{4}{3}\pi r^3$.
- Surface area of a sphere = $4\pi r^2$.
- Area of a circular sector = $\frac{1}{2} r^2 \theta$, when θ is in radians.



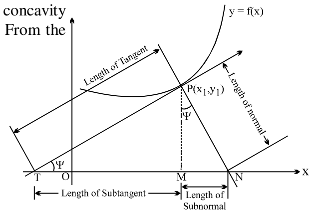
7. SIGNIFICANCE OF THE SIGN OF 2ND ORDER DERIVATIVE AND POINTS OF INFLECTION :

The sign of the 2nd order derivative determines the concavity of the curve. Such points such as C & E on the graph where the concavity of the curve changes are called the points of inflection. From the graph we find that if:

(i) $\frac{d^2y}{dx^2} > 0 \Rightarrow$ concave upwards

(ii) $\frac{d^2y}{dx^2} < 0 \Rightarrow$ concave downwards.

At the point of inflection we find that $\frac{d^2y}{dx^2} = 0$ &



$\int f(x)\,dx = g(x) + c \Leftrightarrow \frac{d}{dx}\{g(x) + c\} = f(x)$, where c is called the **constant of integration**.

2. STANDARD RESULTS :

(i) $\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c \quad n \neq -1$ **(ii)** $\int \frac{dx}{ax+b} = \frac{1}{a}\ln(ax+b) + c$

(iii) $\int e^{ax+b}\,dx = \frac{1}{a} e^{ax+b} + c$ **(iv)** $\int a^{px+q}\,dx = \frac{1}{p}\frac{a^{px+q}}{\ell n a}\ (a > 0) + c$

(v) $\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b) + c$ **(vi)** $\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + c$

(vii) $\int \tan(ax+b)\,dx = \frac{1}{a}\ln\sec(ax+b) + c$ **(viii)** $\int \cot(ax+b)\,dx = \frac{1}{a}\ln\sin(ax+b) + c$

(ix) $\int \sec^2(ax+b)\,dx = \frac{1}{a}\tan(ax+b) + c$ **(x)** $\int \mathrm{cosec}^2(ax+b)\,dx = -\frac{1}{a}\cot(ax+b) + c$

(xi) $\int \sec(ax+b)\ .\ \tan(ax+b)\,dx = \frac{1}{a}\sec(ax+b) + c$

(xii) $\int \mathrm{cosec}(ax+b)\ .\ \cot(ax+b)\,dx = -\frac{1}{a}\mathrm{cosec}(ax+b) + c$

(xiii) $\int \sec x\,dx = \ln(\sec x + \tan x) + c$ **OR** $\ln\tan\left(\frac{\pi}{4} + \frac{x}{2}\right) + c$

(xiv) $\int \mathrm{cosec}\,x\,dx = \ln(\mathrm{cosec}\,x - \cot x) + c$ **OR** $\ln\tan\frac{x}{2} + c$ **OR** $-\ln(\mathrm{cosec}\,x + \cot x)$

(xv) $\int \sinh x\,dx = \cosh x + c$ **(xvi)** $\int \cosh x\,dx = \sinh x + c$ **(xvii)** $\int \mathrm{sech}^2 x\,dx = \tanh x + c$

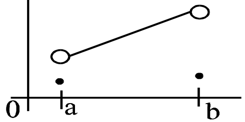
(xviii) $\int \mathrm{cosech}^2 x\,dx = -\coth x + c$ **(xix)** $\int \mathrm{sech}\,x\ .\ \tanh x\,dx = -\mathrm{sech}\,x + c$

(xx) $\int \mathrm{cosech}\,x\ .\ \coth x\,dx = -\mathrm{cosech}\,x + c$ **(xxi)** $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + c$

(xxii) $\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$ **(xxiii)** $\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\frac{x}{a} + c$

(xxxi) $\int e^{ax}.\sin bx\,dx = \dfrac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + c$

(xxxii) $\int e^{ax}.\cos bx\,dx = \dfrac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) + c$



3. TECHNIQUES OF INTEGRATION :

(i) Substitution or change of independent variable .

Integral $I = \int f(x)\,dx$ is changed to $\int f(\phi(t))\,f'(t)\,dt$, by a suitable substitution $x = \phi(t)$ provided the later integral is easier to integrate .

(ii) Integration by part : $\int u.v\,dx = u\int v\,dx - \int\left[\dfrac{du}{dx}.\int v\,dx\right]dx$ where u & v are differentiable function . **Note :** While using integration by parts, choose u & v such that

(a) $\int v\,dx$ is simple & **(b)** $\int\left[\dfrac{du}{dx}.\int v\,dx\right]dx$ is simple to integrate.

This is generally obtained, by keeping the order of u & v as per the order of the letters in **ILATE**, where ; I – Inverse function, L – Logarithmic function ,
A – Algebraic function, T – Trigonometric function & E – Exponential function

(iii) Partial fraction , spiliting a bigger fraction into smaller fraction by known methods .

4. INTEGRALS OF THE TYPE :www.MathsBySuhag.com , www.TekoClasses.com

(i) $\int [f(x)]^n f'(x)\,dx$ **OR** $\int \dfrac{f'(x)}{[f(x)]^n}\,dx$ put $f(x) = t$ & proceed .

(ii) $\int\dfrac{dx}{ax^2+bx+c}$, $\int\dfrac{dx}{\sqrt{ax^2+bx+c}}$, $\int\sqrt{ax^2+bx+c}\,dx$

Express $ax^2 + bx + c$ in the form of perfect square & then apply the standard results .

(iii) $\int\dfrac{px+q}{ax^2+bx+c}\,dx$, $\int\dfrac{px+q}{\sqrt{ax^2+bx+c}}\,dx$.

Express $px + q = A$ (differential co-efficient of denominator) + B .

(iv) $\int e^x[f(x) + f'(x)]\,dx = e^x.f(x) + c$ **(v)** $\int [f(x) + xf'(x)]\,dx = x\,f(x) + c$

(xi) $\int \dfrac{\ldots}{\ell\cos x + m\sin x + n}\,dx$: Express Nr = A(Dr) + B $\dfrac{d}{dx}$(Dr) + c & proceed .

(xii) $\int\dfrac{x^2+1}{x^4+Kx^2+1}\,dx$ **OR** $\int\dfrac{x^2-1}{x^4+Kx^2+1}\,dx$ where K is any constant .

Hint : Divide Nr & Dr by x^2 & proceed .

(xiii) $\int\dfrac{dx}{(ax+b)\sqrt{px+q}}$ & $\int\dfrac{dx}{(ax^2+bx+c)\sqrt{px+q}}$; put $px + q = t^2$.

(xiv) $\int\dfrac{dx}{(ax+b)\sqrt{px^2+qx+r}}$,put $ax + b = \dfrac{1}{t}$; $\int\dfrac{dx}{(ax^2+bx+c)\sqrt{px^2+qx+r}}$, put $x = \dfrac{1}{t}$

(xv) $\int\sqrt{\dfrac{x-\alpha}{\beta-x}}\,dx$ or $\int\sqrt{(x-\alpha)(\beta-x)}$; put $x = \alpha\cos^2\theta + \beta\sin^2\theta$

$\int\sqrt{\dfrac{x-\alpha}{x-\beta}}\,dx$ or $\int\sqrt{(x-\alpha)(x-\beta)}$; put $x = \alpha\sec^2\theta - \beta\tan^2\theta$

$\int\dfrac{dx}{\sqrt{(x-\alpha)(x-\beta)}}$; put $x - \alpha = t^2$ or $x - \beta = t^2$.

DEFINITE INTEGRAL

1. $\int_a^b f(x)\,dx = F(b) - F(a)$ where $\int f(x)\,dx = F(x) + c$

Very Important Note : If $\int_a^b f(x)\,dx = 0 \Rightarrow$ then the equation $f(x) = 0$ has atleast one root lying in (a, b) provided f is a continuous function in (a, b) .

2. Properties Of Definite Integral :www.MathsBySuhag.com , www.TekoClasses.com

P–1 $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$ provided f is same **P – 2** $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

P–3 $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$, where c may lie inside or outside the interval [a, b] . This property to be

P–9 $\int_{ma}^{na} f(x)\,dx = (n-m)\int_0^a f(x)\,dx$ if f(x) is periodic with period 'a' .

P–10 If $f(x) \le \phi(x)$ for $a \le x \le b$ then $\int_a^b f(x)\,dx \le \int_a^b \phi(x)\,dx$

P–11 $\left|\int_a^b f(x)dx\right| \le \int_a^b |f(x)|\,dx$.**P–12** If $f(x) \ge 0$ on the interval [a, b] , then $\int_a^b f(x)\,dx \ge 0$.

3. WALLI'S FORMULA : www.MathsBySuhag.com , www.TekoClasses.com

$$\int_0^{\pi/2} \sin^n x \,.\, \cos^m x\,dx = \frac{[(n-1)(n-3)(n-5)\ldots.1 or 2][(m-1)(m-3)\ldots.1 or 2]}{(m+n)(m+n-2)(m+n-4)\ldots.1 or 2}K$$

Where K $= \frac{\pi}{2}$ if both m and n are even $(m, n \in N)$; $= 1$ otherwise

4. DERIVATIVE OF ANTIDERIVATIVE FUNCTION :

If h(x) & g(x) are differentiable functions of x then ,

$$\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\,dt = f[h(x)]\,.\,h'(x) - f[g(x)]\,.\,g'(x)$$

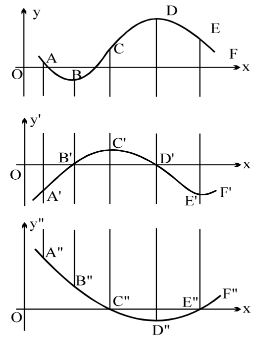
5. DEFINITE INTEGRAL AS LIMIT OF A SUM :

$$\int_a^b f(x)\,dx = \underset{n\to\infty}{\text{Limit}}\; h\,[f(a) + f(a+h) + f(a+2h) + \ldots.. + f(a + \overline{n-1}\,h)]$$

$$= \underset{h\to 0}{\text{Limit}}\; h\sum_{r=0}^{n-1} f(a+rh) \quad \text{where } b-a = nh$$

If $a = 0$ & $b = 1$ then , $\underset{n\to\infty}{\text{Limit}}\; h\sum_{r=0}^{n-1} f(rh) = \int_0^1 f(x)\,dx$; where $nh = 1$ **OR**

15. AREA UNDER CURVE (AUC)



THINGS TO REMEMBER :

1. The area bounded by the curve y = f(x) , the x-axis and the ordinates at x = a & x = b is given by,

$$A = \int_a^b f(x)\,dx = \int_a^b y\,dx.$$

2. If the area is below the x–axis then A is negative. The convention is to consider the magnitude only i.e.

$$A = \left|\int_a^b y\,dx\right| \text{ in this case.}$$

3. Area between the curves y = f (x) & y = g (x) between the ordinates at x = a & x = b is given by,

$$A = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \int_a^b [f(x) - g(x)]\,dx.$$

4. Average value of a function y = f (x) .r.t. *x* over an interval

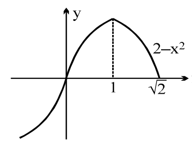
$a \le x \le b$ is defined as : $y(av) = \frac{1}{b-a}\int_a^b f(x)$

5. The area function A_a^x satisfies the differential equation $\frac{dA_a^x}{dx} = f(x)$ with initial condition $A_a^a = 0$.

Note : If F (x) is any integral of f (x) then ,

$A_a^x = \int f(x)\,dx = F(x) + c$ $\quad A_a^a = 0 = F(a) + c \Rightarrow c = -F(a)$

hence $A_a^x = F(x) - F(a)$. Finally by taking x = b we get , $A_a^b = F(b) - F(a)$.



6. CURVE TRACING :

The following outline procedure is to be applied in Sketching the graph of a function y = f (x) which in turn will be extremely useful to quickly and correctly evaluate the area under the curves.

(a) Symmetry : The symmetry of the curve is judged as follows :

DEFINITIONS :

1. An equation that involves independent and dependent variables and the derivatives of the dependent variables is called a **DIFFERENTIAL EQUATION**.www.MathsBySuhag.com , www.TekoClasses.com

2. A differential equation is said to be ordinary , if the differential coefficients have reference to a single independent variable only and it is said to be **PARTIAL** if there are two or more independent variables . We are concerned with ordinary differential equations only. eg. $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$ is a partial differential equation.

3. Finding the unknown function is called **SOLVING OR INTEGRATING** the differential equation . The solution of the differential equation is also called its **PRIMITIVE**, because the differential equation can be regarded as a relation derived from it.

4. The order of a differential equation is the order of the highest differential coefficient occuring in it.

5. The degree of a differential equation which can be written as a polynomial in the derivatives is the degree of the derivative of the highest order occuring in it , after it has been expressed in a form free from radicals & fractions so far as derivatives are concerned, thus the differential equation :

$f(x, y)\left[\frac{d^m y}{dx^m}\right]^p + \phi(x, y)\left[\frac{d^{m-1}(y)}{dx^{m-1}}\right]^q + ... = 0$ is order m & degree p. Note that in the differential equation $e^{y'''} - xy'' + y = 0$ order is three but degree doesn't apply.

6. FORMATION OF A DIFFERENTIAL EQUATION :

If an equation in independent and dependent variables having some arbitrary constant is given , then a differential equation is obtained as follows :

- ☞ Differentiate the given equation w.r.t. the independent variable (say x) as many times as the number of arbitrary constants in it .
- ☞ Eliminate the arbitrary constants .

The eliminant is the required differential equation . Consider forming a differential equation for $y^2 = 4a(x + b)$ where a and b are arbitary constant .

Note : A differential equation represents a family of curves all satisfying some common properties. This can be considered as the geometrical interpretation of the differential equation.

7. GENERAL AND PARTICULAR SOLUTIONS :

The solution of a differential equation which contains a number of independent arbitrary constants equal to the order of the differential equation is called the **GENERAL SOLUTION (OR COMPLETE INTEGRAL OR COMPLETE PRIMITIVE)** . A solution obtainable from the general solution by giving particular values to the constants is called a **PARTICULAR SOLUTION**.

TYPE–2 : $\frac{dy}{dx} = f(ax + by + c)$, $b \neq 0$. To solve this , substitute $t = ax + by + c$. Then the equation reduces to separable type in the variable t and x which can be solved. Consider the example $(x + y)^2 \frac{dy}{dx} = a^2$.

TYPE–3. HOMOGENEOUS EQUATIONS :www.MathsBySuhag.com , www.TekoClasses.com

A differential equation of the form $\frac{dy}{dx} = \frac{f(x,y)}{\phi(x,y)}$ where $f(x , y)$ & $\phi(x , y)$ are homogeneous functions of x & y , and of the same degree , is called **HOMOGENEOUS** . This equation may also be reduced to the form $\frac{dy}{dx} = g\left(\frac{x}{y}\right)$ & is solved by putting $y = vx$ so that the dependent variable y is changed to another variable v, where v is some unknown function, the differential equation is transformed to an equation with variables separable. Consider $\frac{dy}{dx} + \frac{y(x+y)}{x^2} = 0$.

TYPE–4. EQUATIONS REDUCIBLE TO THE HOMOGENEOUS FORM :

If $\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$; where $a_1b_2 - a_2b_1 \neq 0$, i.e. $\frac{a_1}{b_1} \neq \frac{a_2}{b_2}$

then the substitution $x = u + h$, $y = v + k$ transform this equation to a homogeneous type in the new variables u and v where h and k are arbitrary constants to be chosen so as to make the given equation homogeneous which can be solved by the method as given in Type – 3. If

(i) $a_1b_2 - a_2b_1 = 0$, then a substitution $u = a_1x + b_1y$ transforms the differential equation to an equation with variables separable. and

(ii) $b_1 + a_2 = 0$, then a simple cross multiplication and substituting $d(xy)$ for $x\,dy + y\,dx$ & integrating term by term yields the result easily.

Consider $\frac{dy}{dx} = \frac{x - 2y + 5}{2x + y - 1}$; $\frac{dy}{dx} = \frac{2x + 3y - 1}{4x + 6y - 5}$ & $\frac{dy}{dx} = \frac{2x - y + 1}{6x - 5y + 4}$

(iii) In an equation of the form : $y f(xy)\,dx + x g(xy)dy = 0$ the variables can be separated by the substitution $xy = v$.

IMPORTANT NOTE :

(a) The function $f(x , y)$ is said to be a homogeneous function of degree n if for any real number t $(\neq 0)$, we have $f(tx , ty) = t^n f(x , y)$.

TYPE – 5. LINEAR DIFFERENTIAL EQUATIONS OF FIRST ORDER :

The most general form of a linear differential equations of first order is $\frac{dy}{dx} + Py = Q$, where

P& Q are functions of x . To solve such an equation multiply both sides by $e^{\int P dx}$.

NOTE : (1) The factor $e^{\int P dx}$ on multiplying by which the left hand side of the differential equation becomes the differential coefficient of some function of x & y , is called integrating factor of the differential equation popularly abbreviated as I. F.

(2) It is very important to remember that on multiplying by the integrating factor , the left hand side becomes the derivative of the product of y and the I. F.

(3) Some times a given differential equation becomes linear if we take y as the independent variable and x as the dependent variable. e.g. the equation ;

$(x + y + 1)\frac{dy}{dx} = y^2 + 3$ can be written as $(y^2 + 3)\frac{dx}{dy} = x + y + 1$ which is a linear differential equation.

TYPE–6. EQUATIONS REDUCIBLE TO LINEAR FORM :

The equation $\frac{dy}{dx} + py = Q \cdot y^n$ where P & Q functions of x , is reducible to the linear form by dividing it by y^n & then substituting $y^{-n+1} = Z$. Its solution can be obtained as in **Type–5**. Consider the example $(x^3y^2 + xy)\,dx = dy$.

The equation $\frac{dy}{dx} + Py = Q \cdot y^n$ is called **BERNOULI'S EQUATION**.

9. TRAJECTORIES :

Suppose we are given the family of plane curves. $\Phi(x, y, a) = 0$ depending on a single parameter a. A curve making at each of its points a fixed angle α with the curve of the family passing through that point is called an ***isogonal trajectory*** of that family ; if in particular $\alpha = \pi/2$, then it is called an ***orthogonal trajectory***.

Orthogonal trajectories : We set up the differential equation of the given family of curves. Let it be of the form $F(x, y, y') = 0$ The differential equation of the orthogonal trajectories is of the form $F\left(x, y, -\frac{1}{y'}\right) =$

(xiii) $d\left(\frac{e^y}{x}\right) = \frac{x e^y\,dy - e^y\,dx}{x^2}$

17. STRAIGHT LINES & PAIR OF STRAIGHT LINES

1. DISTANCE FORMULA :

The distance between the points $A(x_1,y_1)$ and $B(x_2,y_2)$ is $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$.

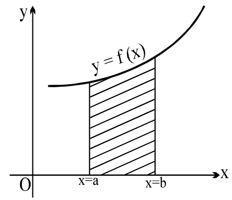
2. SECTION FORMULA :

If P(x , y) divides the line joining $A(x_1 , y_1)$ & $B(x_2 , y_2)$ in the ratio m : n, then ;

$x = \frac{mx_2+nx_1}{m+n}$; $y = \frac{my_2+ny_1}{m+n}$ If $\frac{m}{n}$ is positive, the division is internal, but if $\frac{m}{n}$ is negative, the division is external .

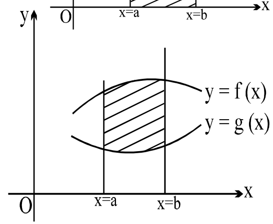
Note : If P divides AB internally in the ratio m : n & Q divides AB externally in the ratio m : n then P & Q are said to be harmonic conjugate of each other w.r.t. AB.

Mathematically ; $\frac{2}{AB} = \frac{1}{AP} + \frac{1}{AQ}$ i.e. AP, AB & AQ are in H.P.



3. CENTROID AND INCENTRE :

If $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ are the vertices of triangle ABC, whose sides BC, CA, AB are of lengths a, b, c respectively, then the coordinates of the centroid are : $\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$ & the coordinates of the incentre are : $\left(\frac{ax_1+bx_2+cx_3}{a+b+c}, \frac{ay_1+by_2+cy_3}{a+b+c}\right)$ Note that incentre divides the angle bisectors in the ratio $(b+c) : a$; $(c+a) : b$ & $(a+b) : c$.



REMEMBER :(i) Orthocentre , Centroid & circumcentre are always collinear & centroid divides the line joining orthocentre & cercumcentre in the ratio 2 : 1 .

(ii) In an isosceles triangle G, O, I & C lie on the same line .

$\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r$ (say). Where 'r' is the distance of any point (x , y) on the line from the fixed point (x_1, y_1) on the line. r is positive if the point (x, y) is on the right of (x_1, y_1) and negative if (x, y) lies on the left of (x_1, y_1) .www.MathsBySuhag.com , www.TekoClasses.com

(iv) **Two point form :** $y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$ is the equation of a straight line which passes through the points (x_1, y_1) & (x_2, y_2) .

(v) **Intercept form :** $\frac{x}{a} + \frac{y}{b} = 1$ is the equation of a straight line which makes intercepts a & b on OX & OY respectively .

(vi) **Perpendicular form :** $x\cos\alpha + y\sin\alpha = p$ is the equation of the straight line where the length of the perpendicular from the origin O on the line is p and this perpendicular makes angle α with positive side of x–axis .

(vii) **General Form :** $ax + by + c = 0$ is the equation of a straight line in the general form

7. **POSITION OF THE POINT (x_1, y_1) RELATIVE TO THE LINE $ax + by + c = 0$:** If $ax_1 + by_1 + c$ is of the same sign as c, then the point (x_1, y_1) lie on the origin side of $ax + by + c = 0$. But if the sign of $ax_1 + by_1 + c$ is opposite to that of c, the point (x_1, y_1) will lie on the non-origin side of $ax + by + c = 0$.

8. **THE RATIO IN WHICH A GIVEN LINE DIVIDES THE LINE SEGMENT JOINING TWO POINTS :**

Let the given line $ax + by + c = 0$ divide the line segment joining $A(x_1, y_1)$ & $B(x_2, y_2)$ in the ratio m : n, then $\frac{m}{n} = -\frac{ax_1 + by_1 + c}{ax_2 + by_2 + c}$. If A & B are on the same side of the given line then $\frac{m}{n}$ is negative but if A & B are on opposite sides of the given line , then $\frac{m}{n}$ is positive

9. **LENGTH OF PERPENDICULAR FROM A POINT ON A LINE :**

The length of perpendicular from $P(x_1, y_1)$ on $ax + by + c = 0$ is $\left|\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|$.

10. **ANGLE BETWEEN TWO STRAIGHT LINES IN TERMS OF THEIR SLOPES :**

If m_1 & m_2 are the slopes of two intersecting straight lines $(m_1 m_2 \neq -1)$ & θ is the acute angle

bounded by the lines $y = m_1x + c_1$, $y = m_1x + c_2$ and $y = m_2x + d_1$, $y = m_2x + d_2$ is given by $\left|\frac{(c_1 - c_2)(d_1 - d_2)}{m_1 - m_2}\right|$.

12. **PERPENDICULAR LINES :**

(i) When two lines of slopes m_1 & m_2 are at right angles, the product of their slopes is -1, i.e. $m_1 m_2 = -1$. Thus any line perpendicular to $ax + by + c = 0$ is of the form $bx - ay + k = 0$, where k is any parameter.www.MathsBySuhag.com , www.TekoClasses.com

(ii) St. lines $ax + by + c = 0$ & $a'x + b'y + c' = 0$ are right angles if & only if $aa' + bb' = 0$.

13. Equations of straight lines through (x_1 , y_1) making angle α with $y = mx + c$ are:
$(y - y_1) = \tan(\theta - \alpha)(x - x_1)$ & $(y - y_1) = \tan(\theta + \alpha)(x - x_1)$, where $\tan\theta = m$.

14. **CONDITION OF CONCURRENCY :**

Three lines $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$ & $a_3x + b_3y + c_3 = 0$ are concurrent if $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$. **Alternatively :** If three constants A, B & C can be found such that $A(a_1x + b_1y + c_1) + B(a_2x + b_2y + c_2) + C(a_3x + b_3y + c_3) \equiv 0$, then the three straight lines are concurrent.

15. **AREA OF A TRIANGLE :** If (x_i, y_i), i = 1, 2, 3 are the vertices of a triangle, then its area is equal to $\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$, provided the vertices are considered in the counter clockwise sense. The above formula will give a (–) ve area if the vertices (x_i, y_i) , i = 1, 2, 3 are placed in the clockwise sense.

16. **CONDITION OF COLLINEARITY OF THREE POINTS–(AREA FORM):**

The points (x_i , y_i) , i = 1 , 2 , 3 are collinear if $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$.

17. **THE EQUATION OF A FAMILY OF STRAIGHT LINES PASSING THROUGH THE POINTS OF INTERSECTION OF TWO GIVEN LINES:**

The equation of a family of lines passing through the point of intersection of $a_1x + b_1y + c_1 = 0$ & $a_2x + b_2y + c_2 = 0$ is given by $(a_1x + b_1y + c_1) + k(a_2x + b_2y + c_2) = 0$, where k is an

If $|\tan\theta| > 1$, then we get the bisector to be the obtuse angle bisector .

(iii) To discriminate between the bisector of the angle containing the origin & that of the angle not containing the origin. Rewrite the equations , $ax + by + c = 0$ & $a'x + b'y + c' = 0$ such that the constant terms c, c' are positive. Then; $\frac{ax+by+c}{\sqrt{a^2+b^2}} = + \frac{a'x+b'y+c'}{\sqrt{a'^2+b'^2}}$ gives the equation of the bisector of the angle containing the origin & $\frac{ax+by+c}{\sqrt{a^2+b^2}} = - \frac{a'x+b'y+c'}{\sqrt{a'^2+b'^2}}$ gives the equation of the bisector of the angle not containing the origin.www.MathsBySuhag.com , www.TekoClasses.com

(iv) To discriminate between acute angle bisector & obtuse angle bisector proceed as follows
Write $ax + by + c = 0$ & $a'x + b'y + c' = 0$ such that constant terms are positive .
If $aa' + bb' < 0$, then the angle between the lines that contains the origin is acute and the equation of the bisector of this acute angle is $\frac{ax+by+c}{\sqrt{a^2+b^2}} = + \frac{a'x+b'y+c'}{\sqrt{a'^2+b'^2}}$

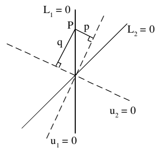
therefore $\frac{ax+by+c}{\sqrt{a^2+b^2}} = - \frac{a'x+b'y+c'}{\sqrt{a'^2+b'^2}}$ is the equation of other bisector.

If, however , $aa' + bb' > 0$, then the angle between the lines that contains the origin is obtuse & the equation of the bisector of this obtuse angle is:

$\frac{ax+by+c}{\sqrt{a^2+b^2}} = + \frac{a'x+b'y+c'}{\sqrt{a'^2+b'^2}}$; therefore $\frac{ax+by+c}{\sqrt{a^2+b^2}} = - \frac{a'x+b'y+c'}{\sqrt{a'^2+b'^2}}$

is the equation of other bisector.

(v) Another way of identifying an acute and obtuse angle bisector is as follows :
Let $L_1 = 0$ & $L_2 = 0$ are the given lines & $u_1 = 0$ and $u_2 = 0$ are the bisectors between $L_1 = 0$ & $L_2 = 0$. Take a point P on any one of the lines $L_1 = 0$ or $L_2 = 0$ and drop perpendicular on $u_1 = 0$ & $u_2 = 0$ as shown. If ,



$|p| < |q| \Rightarrow u_1$ is the acute angle bisector .
$|p| > |q| \Rightarrow u_1$ is the obtuse angle bisector .
$|p| = |q| \Rightarrow$ the lines L_1 & L_2 are perpendicular .

Note : Equation of straight lines passing through $P(x_1, y_1)$ & equally inclined with the lines $a_1x + b_1y + c_1 = 0$ & $a_2x + b_2y + c_2 = 0$ are those which are parallel to the bisectors between these two lines & passing through the point P .

19. A PAIR OF STRAIGHT LINES THROUGH ORIGIN :

(i) A homogeneous equation of degree two of the type $ax^2 + 2hxy + by^2 = 0$ always represents a pair of

(i) $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of straight lines if:

$abc + 2fgh - af^2 - bg^2 - ch^2 = 0$, i.e. if $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$.

(ii) The angle θ between the two lines representing by a general equation is the same as that between the two lines represented by its homogeneous part only .

21. The joint equation of a pair of straight lines joining origin to the points of intersection of the line given by $lx + my + n = 0$ .... **(i)** &
the 2nd degree curve : $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ .... **(ii)**

is $ax^2 + 2hxy + by^2 + 2gx\left(\frac{lx+my}{-n}\right) + 2fy\left(\frac{lx+my}{-n}\right) + c\left(\frac{lx+my}{-n}\right)^2 = 0$ ...... **(iii)**

(iii) is obtained by homogenizing (ii) with the help of **(i)**, by writing **(i)** in the form: $\left(\frac{lx+my}{-n}\right) = 1$.

22. The equation to the straight lines bisecting the angle between the straight lines,

$ax^2 + 2hxy + by^2 = 0$ is $\frac{x^2 - y^2}{a - b} = \frac{xy}{h}$.www.MathsBySuhag.com , www.TekoClasses.com

23. The product of the perpendiculars, dropped from (x_1, y_1) to the pair of lines represented by the equation, $ax^2 + 2hxy + by^2 = 0$ is $\frac{ax_1^2 + 2hx_1y_1 + by_1^2}{\sqrt{(a-b)^2 + 4h^2}}$.

24. Any second degree curve through the four point of intersection of $f(xy) = 0$ & $xy = 0$ is given by $f(xy) + \lambda xy = 0$ where $f(xy) = 0$ is also a second degree curve.

18. CIRCLE

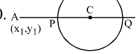
STANDARD RESULTS :

1. EQUATION OF A CIRCLE IN VARIOUS FORM :

(a) The circle with centre(h, k) & radius'r'has the equation ;$(x - h)^2 + (y - k)^2 = r^2$.

(b) The general equation of a circle is $x^2 + y^2 + 2gx + 2fy + c = 0$ with centre as :
$(-g, -f)$ & radius $= \sqrt{g^2 + f^2 - c}$.

The point (x_1, y_1) is inside, on or outside the circle $x^2 + y^2 + 2gx + 2fy + c = 0$. according as $x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c \Leftrightarrow 0$.



Note : The greatest & the least distance of a point A from a circle with centre C & radius r is AC + r & AC − r respectively.

4. LINE & A CIRCLE :

Let L = 0 be a line & S = 0 be a circle. If r is the radius of the circle & p is the length of the perpendicular from the centre on the line, then :www.MathsBySuhag.com , www.TekoClasses.com

(i) $p > r \Leftrightarrow$ the line does not meet the circle i. e. passes out side the circle.

(ii) $p = r \Leftrightarrow$ the line touches the circle.

(iii) $p < r \Leftrightarrow$ the line is a secant of the circle.

(iv) $p = 0 \Rightarrow$ the line is a diameter of the circle.

5. PARAMETRIC EQUATIONS OF A CIRCLE :

The parametric equations of $(x - h)^2 + (y - k)^2 = r^2$ are :

$x = h + r\cos\theta$; $y = k + r\sin\theta$; $-\pi < \theta \le \pi$ where (h, k) is the centre,

r is the radius & θ is a parameter. Note that equation of a straight line joining two point α & β on the circle $x^2 + y^2 = a^2$ is $x\cos\frac{\alpha+\beta}{2} + y\sin\frac{\alpha+\beta}{2} = a\cos\frac{\alpha-\beta}{2}$.

6. TANGENT & NORMAL :

(a) The equation of the tangent to the circle $x^2 + y^2 = a^2$ at its point (x_1, y_1) is, $xx_1 + yy_1 = a^2$. Hence equation of a tangent at $(a\cos\alpha, a\sin\alpha)$ is ; $x\cos\alpha + y\sin\alpha = a$. The point of intersection of the tangents at the points $P(\alpha)$ and $Q(\beta)$ is

$$\left(\frac{a\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}, \frac{a\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}\right).$$

(b) The equation of the tangent to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ at its point (x_1, y_1) is $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$.

(c) $y = mx + c$ is always a tangent to the circle $x^2 + y^2 = a^2$ if $c^2 = a^2(1 + m^2)$ and the point of contact is $\left(-\frac{a^2m}{c}, \frac{a^2}{c}\right)$.

(d) If a line is normal/orthogonal to a circle then it must pass through the centre of the circle. Using this fact normal to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ at (x_1, y_1) is

$$y - y_1 = \frac{y_1+f}{x_1+g}(x - x_1).$$

(f) Equation of circle circumscribing a quadrilateral whose side in order are represented by the lines $L_1 = 0, L_2 = 0, L_3 = 0$ & $L_4 = 0$ is $L_1L_3 + \lambda L_2L_4 = 0$ provided co-efficient of x^2 = co-efficient of y^2 and co-efficient of $xy = 0$.

8. LENGTH OF A TANGENT AND POWER OF A POINT :

The length of a tangent from an external point (x_1, y_1) to the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ is given by $L = \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2f_1y + c} = \sqrt{S_1}$. Square of length of the tangent from the point P is also called **THE POWER OF POINT** w.r.t. a circle. Power of a point remains constant w.r.t. a circle. **Note that** : power of a point P is positive, negative or zero according as the point 'P' is outside, inside or on the circle respectively.

9. DIRECTOR CIRCLE :

The locus of the point of intersection of two perpendicular tangents is called the **DIRECTOR CIRCLE** of the given circle. The director circle of a circle is the concentric circle having radius equal to $\sqrt{2}$ times the original circle.

10. EQUATION OF THE CHORD WITH A GIVEN MIDDLE POINT :

The equation of the chord of the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ in terms of its mid point $M(x_1, y_1)$ is $y - y_1 = -\frac{x_1+g}{y_1+f}(x - x_1)$. This on simplication can be put in the form $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$ which is designated by $T = S_1$. **Note that** : the shortest chord of a circle passing through a point 'M' inside the circle, is one chord whose middle point is M.

11. CHORD OF CONTACT :

If two tangents PT_1 & PT_2 are drawn from the point $P(x_1, y_1)$ to the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$, then the equation of the chord of contact T_1T_2 is : $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$.

REMEMBER : **(a)** Chord of contact exists only if the point 'P' is not inside .

(b) Length of chord of contact $T_1T_2 = \frac{2LR}{\sqrt{R^2+L^2}}$.

(c) Area of the triangle formed by the pair of the tangents & its chord of contact $= \frac{RL^3}{R^2+L^2}$ Where R is the radius of the circle & L is the length of the tangent from (x_1, y_1) on $S = 0$.

(iii) Pole of a given line $Ax + By + C = 0$ w.r.t. any circle $x^2 + y^2 = a^2$ is $\left(-\frac{Aa}{C}, -\frac{Ba}{C}\right)$.

(iv) If the polar of a point P pass through a point Q, then the polar of Q passes through P.

(v) Two lines L_1 & L_2 are conjugate of each other if Pole of L_1 lies on L_2 & vice versa Similarly two points P & Q are said to be conjugate of each other if the polar of P passes through Q & vice-versa.

13. COMMON TANGENTS TO TWO CIRCLES :

(i) Where the two circles neither intersect nor touch each other, there are FOUR common tangents, two of them are transverse & the others are direct common tangents.

(ii) When they intersect there are two common tangents, both of them being direct.

(iii) When they touch each other :www.MathsBySuhag.com , www.TekoClasses.com

(a) **Externally :** there are three common tangents, two direct and one is the tangent at the point of contact .

(b) **Internally :** only one common tangent possible at their point of contact.

(iv) Length of an external common tangent & internal common tangent to the two circles is given by:

$L_{ext} = \sqrt{d^2 - (r_1 - r_2)^2}$ & $L_{int} = \sqrt{d^2 - (r_1 + r_2)^2}$.

Where d = distance between the centres of the two circles . r_1 & r_2 are the radii of the 2 circles.

(v) The direct common tangents meet at a point which divides the line joining centre of circles externally in the ratio of their radii.

Transverse common tangents meet at a point which divides the line joining centre of circles internally in the ratio of their radii.

14. RADICAL AXIS & RADICAL CENTRE :

The radical axis of two circles is the locus of points whose powers w.r.t. the two circles are equal. The equation of radical axis of the two circles $S_1 = 0$ & $S_2 = 0$ is given ;

$S_1 - S_2 = 0$ i.e. $2(g_1 - g_2)x + 2(f_1 - f_2)y + (c_1 - c_2) = 0$.

NOTE THAT :

(a) If two circles intersect, then the radical axis is the common chord of the two circles.

(b) If two circles touch each other then the radical axis is the common tangent of the two circles at the common point of contact.

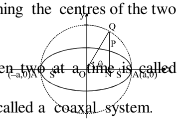
(c) Radical axis is always perpendicular to the line joining the centres of the 2circles.

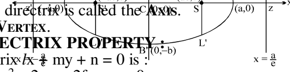
(d) Radical axis need not always pass through the mid point of the line joining the centres of the two circles.

(e) Radical axis bisects a common tangent between the two circles.

(f) The common point of intersection of the radical axes of three circles taken two at a time is called the radical centre of three circles.

(g) A system of circles , every two which have the same radical axis, is called a coaxal system.





☞ The line passing through the focus & perpendicular to the directrix is called the **Axis**.

☞ A point of intersection of a conic with its axis is called a **Vertex**.

2. GENERAL EQUATION OF A CONIC : FOCAL DIRECTRIX PROPERTY :

The general equation of a conic with focus (p, q) & directrix $lx + my + n = 0$ is :

$(l^2 + m^2)[(x - p)^2 + (y - q)^2] = e^2(lx + my + n)^2 \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

3. DISTINGUISHING BETWEEN THE CONIC :

The nature of the conic section depends upon the position of the focus S w.r.t. the directrix & also upon the value of the eccentricity e. Two different cases arise.

Case (I) : When The Focus Lies On The Directrix.

In this case $D \equiv abc + 2fgh - af^2 - bg^2 - ch^2 = 0$ & the general equation of a conic represents a pair of straight lines if :www.MathsBySuhag.com , www.TekoClasses.com

$e > 1$ the lines will be real & distinct intersecting at S.

$e = 1$ the lines will coincident.

$e < 1$ the lines will be imaginary.

Case (II) : When The Focus Does Not Lie On Directrix.

a parabola	an ellipse	a hyperbola	rectangular hyperbola
$e = 1 ; D \neq 0$,	$0 < e < 1 ; D \neq 0$;	$e > 1 ; D \neq 0$;	$e > 1 ; D \neq 0$
$h^2 = ab$	$h^2 < ab$	$h^2 > ab$	$h^2 > ab ; a + b = 0$

4. PARABOLA : DEFINITION :

A parabola is the locus of a point which moves in a plane, such that its distance from a fixed point (focus) is equal to its perpendicular distance from a fixed straight line (directrix).

Standard equation of a parabola is $y^2 = 4ax$. For this parabola :

(i) Vertex is (0, 0) (ii) focus is (a, 0) (iii) Axis is $y = 0$ (iv) Directrix is $x + a = 0$

FOCAL DISTANCE : The distance of a point on the parabola from the focus is called the **Focal Distance Of The Point.**

FOCAL CHORD :

A chord of the parabola, which passes through the focus is called a **Focal Chord.**

DOUBLE ORDINATE : A chord of the parabola perpendicular to the axis of the symmetry is called a **Double Ordinate.**

LATUS RECTUM :

A double ordinate passing through the focus or a focal chord perpendicular to the axis of parabola is called the **Latus Rectum.** For $y^2 = 4ax$.

■Length of the latus rectum = 4a. ■ends of the latus rectum are L(a, 2a) & L' (a, − 2a).

Note that: **(i)** Perpendicular distance from focus on directrix = half the latus rectum.

(ii) Vertex is middle point of the focus & the point of intersection of directrix & axis.

(iii) Two parabolas are laid to be equal if they have the same latus rectum.

Four standard forms of the parabola are $y^2 = 4ax$; $y^2 = -4ax$; $x^2 = 4ay$; $x^2 = -4ay$

5. POSITION OF A POINT RELATIVE TO A PARABOLA :

The point (x_1, y_1) lies outside, on or inside the parabola $y^2 = 4ax$ according as the expression $y_1^2 - 4ax_1$ is positive, zero or negative.

Note : Point of intersection of normals at t_1 & t_2 are, $a(t_1^2 + t_2^2 + t_1t_2 + 2)$; $-a\,t_1\,t_2\,(t_1 + t_2)$.

11. THREE VERY IMPORTANT RESULTS :

(a) If t_1 & t_2 are the ends of a focal chord of the parabola $y^2 = 4ax$ then $t_1t_2 = -1$. Hence the co-ordinates at the extremities of a focal chord can be taken as $(at^2, 2at)$ & $\left(\frac{a}{t^2}, -\frac{2a}{t}\right)$.

(b) If the normals to the parabola $y^2 = 4ax$ at the point t_1, meets the parabola again at the point t_2, then $t_2 = -\left(t_1 + \frac{2}{t_1}\right)$.

(c) If the normals to the parabola $y^2 = 4ax$ at the points t_1 & t_2 intersect again on the parabola at the point 't_3' then $t_1 t_2 = 2$; $t_3 = -(t_1 + t_2)$ and the line joining t_1 & t_2 passes through a fixed point $(-2a, 0)$.

General Note :

(i) Length of subtangent at any point P(x, y) on the parabola $y^2 = 4ax$ equals twice the abscissa of the point P. Note that the subtangent is bisected at the vertex.

(ii) Length of subnormal is constant for all points on the parabola & is equal to the semi latus rectum.

(iii) If a family of straight lines can be represented by an equation $\lambda^2P + \lambda Q + R = 0$ where λ is a parameter and P, Q, R are linear functions of x and y then the family of lines will be tangent to the curve $Q^2 = 4\,PR$.

12. The equation to the pair of tangents which can be drawn from any point (x_1, y_1) to the parabola $y^2 = 4ax$ is given by : $SS_1 = T^2$ where :
$S \equiv y^2 - 4ax$; $S_1 = y_1^2 - 4ax_1$; $T \equiv y\,y_1 - 2a(x + x_1)$.

13. DIRECTOR CIRCLE :
Locus of the point of intersection of the perpendicular tangents to the parabola $y^2 = 4ax$ is called the **Director Circle.** It's equation is $x + a = 0$ which is parabola's own directrix.

14. CHORD OF CONTACT : www.MathsBySuhag.com , www.TekoClasses.com
Equation to the chord of contact of tangents drawn from a point $P(x_1, y_1)$ is $yy_1 = 2a\,(x + x_1)$.
Remember that the area of the triangle formed by the tangents from the point (x_1, y_1) & the chord of contact is $(y_1^2 - 4ax_1)^{3/2} \div 2a$. Also note that the chord of contact exists only if the point P is not inside.

15. POLAR & POLE :

(i) Equation of the Polar of the point $P(x_1, y_1)$ w.r.t. the parabola $y^2 = 4ax$ is
$$y\,y_1 = 2a(x + x_1)$$

(ii) The pole of the line $lx + my + n = 0$ w.r.t. the parabola $y^2 = 4ax$ is $\left(\frac{n}{l}, -\frac{2am}{l}\right)$.

Note:

(i) The polar of the focus of the parabola is the directrix.

(ii) When the point (x_1, y_1) lies without the parabola the equation to its polar is the same as the equation to the chord of contact of tangents drawn from (x_1, y_1) when (x_1, y_1) is on the parabola the polar is the same as the tangent at the point.

(iii) If the polar of a point P passes through the point Q, then the polar of Q goes through P.

(iv) Two straight lines are said to be conjugated to each other w.r.t. a parabola when the pole of one lies on the other.

(v) Polar of a given point P w.r.t. any Conic is the locus of the harmonic conjugate of P w.r.t. the two points is

the **focus.**www.MathsBySuhag.com , www.TekoClasses.com

(c) The tangents at the extremities of a focal chord intersect at right angles on the **directrix**, and hence a circle on any focal chord as diameter touches the directrix. Also a circle on any focal radii of a point P $(at^2, 2at)$ as diameter touches the tangent at the vertex and intercepts a chord of length $a\sqrt{1 + t^2}$ on a normal at the point P.

(d) Any tangent to a parabola & the perpendicular on it from the focus meet on the tangtent at the vertex.

(e) If the tangents at P and Q meet in T, then :■ TP and TQ subtend equal angles at the focus S. ■ $ST^2 = SP.\,SQ$ &■ The triangles SPT and STQ are similar.

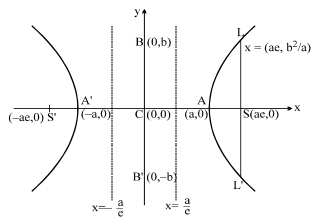
(f) Tangents and Normals at the extremities of the latus rectum of a parabola $y^2 = 4ax$ constitute a square, their points of intersection being $(-a, 0)$ & $(3a, 0)$.

(g) Semi latus rectum of the parabola $y^2 = 4ax$, is the harmonic mean between segments of any focal chord of the parabola is ; $2a = \frac{2bc}{b+c}$ **i.e.** $\frac{1}{b} + \frac{1}{c} = \frac{1}{a}$.

(h) The circle circumscribing the triangle formed by any three tangents to a parabola passes through the focus.

(i) The orthocentre of any triangle formed by three tangents to a parabola $y^2 = 4ax$ lies on the directrix & has the co-ordinates $-a, a(t_1 + t_2 + t_3 + t_1t_2t_3)$.

(j) The area of the triangle formed by three points on a parabola is twice the area of the triangle formed by the tangents at these points.



(k) If normal drawn to a parabola passes through a point P(h, k) then
$k = mh - 2am - am^3$ i.e. $am^3 + m(2a - h) + k = 0$.
Then gives $m_1 + m_2 + m_3 = 0$; $m_1m_2 + m_2m_3 + m_3m_1 = \frac{2a-h}{a}$; $m_1\,m_2\,m_3 = -\frac{k}{a}$.
where m_1, m_2, & m_3 are the slopes of the three concurrent normals. Note that the algebraic sum of the: ■ slopes of the three concurrent normals is zero.
■ ordinates of the three conormal points on the parabola is zero.
■ Centroid of the Δ formed by three co-normal points lies on the x-axis.

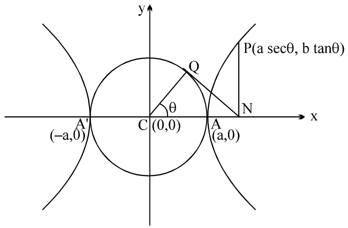
(*l*) A circle circumscribing the triangle formed by three co–normal points passes through the vertex of the parabola and its equation is, $2(x^2 + y^2) - 2(h + 2a)x - ky = 0$

***Suggested problems from S.L.Loney*: Exercise-25** (Q.5, 10, 13, 14, 18, 21), **Exercise-26 (Important)** (Q.4, 6, 7, 16, 17, 20, 22, 26, 27, 28, 34, 38), **Exercise-27** (Q.4, 7), **Exercise-28** (Q.2, 7, 11, 14, 17, 23), **Exercise-29** (Q.7, 8, 10, 19, 21, 24, 26, 27), **Exercise-30** (2, 3, 13, 18, 20, 21, 22, 25, 26, 30)
***Note*:** Refer to the figure on Pg.175 if necessary.

a 2 b 2 2

PRINCIPAL AXIS :
The major & minor axis together are called **Principal Axis** of the ellipse.

CENTRE :
The point which bisects every chord of the conic drawn through it is called the **centre** of the conic. $C \equiv (0, 0)$ the origin is the centre of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.



DIAMETER :
A chord of the conic which passes through the centre is called a **diameter** of the conic.

FOCAL CHORD : A chord which passes through a focus is called a **focal chord**.

DOUBLE ORDINATE :
A chord perpendicular to the major axis is called a **double ordinate.**

LATUS RECTUM : The focal chord perpendicular to the major axis is called the **latus rectum**. Length of latus rectum **(LL′)** = $\frac{2b^2}{a}=\frac{(minor\ axis)^2}{major\ axis}=2a(1-e^2)=$ 2e (distance from focus to the corresponding directrix)

NOTE :

(i) The sum of the focal distances of any point on the ellipse is equal to the major Axis. Hence distance of focus from the extremity of a minor axis is equal to semi major axis. **i.e. BS = CA.**

(ii) If the equation of the ellipse is given as $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ & nothing is mentioned then the rule is to assume that $a > b$.

2. POSITION OF A POINT w.r.t. AN ELLIPSE :

The point $P(x_1, y_1)$ lies outside, inside or on the ellipse according as ; $\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1$ > < or = 0.

3. AUXILIARY CIRCLE / ECCENTRIC ANGLE :
A circle described on major axis as diameter is called the **auxiliary circle**.
Let Q be a point on the auxiliary circle $x^2 + y^2 = a^2$ such that QP produced is perpendicular to the x-axis then P & Q are called as the CORRESPONDING

6. TANGENTS :

(i) $\frac{x\,x_1}{a^2}+\frac{y\,y_1}{b^2}=1$ is tangent to the ellipse at (x_1, y_1).

Note : The figure formed by the tangents at the extremities of latus rectum is rhoubus of area

(ii) $y = mx \pm \sqrt{a^2m^2+b^2}$ is tangent to the ellipse for all values of m.
Note that there are two tangents to the ellipse having the same m, i.e. there are two tangents parallel to any given direction.

(iii) $\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$ is tangent to the ellipse at the point $(a \cos \theta, b \sin \theta)$.

(iv) The eccentric angles of point of contact of two parallel tangents differ by π. Conversely if the difference between the eccentric angles of two points is p then the tangents at these points are parallel.

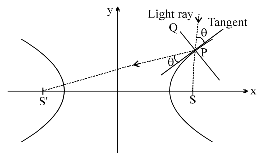
(v) Point of intersection of the tangents at the point α & β is $a\frac{\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}, b\frac{\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}$.

7. NORMALS : www.MathsBySuhag.com , www.TekoClasses.com

(i) Equation of the normal at (x_1, y_1) is $\frac{a^2x}{x_1}-\frac{b^2y}{y_1}=a^2-b^2=a^2e^2$.

(ii) Equation of the normal at the point $(a\cos\theta, b\sin\theta)$ is ; $ax\sec\theta - by\,\mathrm{cosec}\,\theta = (a^2 - b^2)$.

(iii) Equation of a normal in terms of its slope 'm' is $y = mx - \frac{(a^2-b^2)m}{\sqrt{a^2+b^2m^2}}$.



8. DIRECTOR CIRCLE :
Locus of the point of intersection of the tangents which meet at right angles is called the **Director Circle**. The equation to this locus is $x^2 + y^2 = a^2 + b^2$ i.e. a circle whose centre is the centre of the ellipse & whose radius is the length of the line joining the ends of the major & minor axis.

9. Chord of contact, pair of tangents, chord with a given middle point, pole & polar are to be interpreted as they are in parabola.

10. DIAMETER :
The locus of the middle points of a system of parallel chords with slope 'm' of an ellipse is a straight line

(v) locus of the mid point of Gg is another ellipse having the same eccentricity as that of the original ellipse. [where S and S′ are the focii of the ellipse and T is the point where tangent at P meet the major axis]

H – 4 The tangent & normal at a point P on the ellipse bisect the external & internal angles between the focal distances of P. This refers to the well known reflection property of the ellipse which states that rays from one focus are reflected through other focus & vice–versa. Hence we can deduce that the straight lines joining each focus to the foot of the perpendicular from the other focus upon the tangent at any point P meet on the normal PG and bisects it where G is the point where normal at P meets the major axis.

H – 5 The portion of the tangent to an ellipse between the point of contact & the directrix subtends a right angle at the corresponding focus.www.MathsBySuhag.com , www.TekoClasses.com

H – 6 The circle on any focal distance as diameter touches the auxiliary circle.

H – 7 Perpendiculars from the centre upon all chords which join the ends of any perpendicular diameters of the ellipse are of constant length.

H – 8 If the tangent at the point P of a standard ellipse meets the axis in T and t and CY is the perpendicular on it from the centre then,

(i) T t. PY = $a^2 - b^2$ and (ii) least value of Tt is a + b.

HYPERBOLA

The **Hyperbola** is a conic whose eccentricity is greater than unity. (e > 1).

1. STANDARD EQUATION & DEFINITION(S)

Standard equation of the hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Where $b^2 = a^2 (e^2 - 1)$

or $a^2 e^2 = a^2 + b^2$ i.e. $e^2 = 1 + \frac{b^2}{a^2}$

$= 1 + \left(\frac{C.A}{T.A}\right)^2$

FOCI :

$S \equiv (ae, 0)$ & $S' \equiv (-ae, 0)$.

EQUATIONS OF DIRECTRICES : $x = \frac{a}{e}$ & $x = -\frac{a}{e}$.

VERTICES : $A \equiv (a, 0)$ & $A' \equiv (-a, 0)$. ***l* (Latus rectum)** $= \frac{2b^2}{a} = \frac{(C.A.)^2}{T.A.} = 2a(e^2 - 1)$.

(b) The foci of a hyperbola and its conjugate are concyclic and form the vertices of a square.

(c) Two hyperbolas are said to be similiar if they have the same eccentricity.

4. RECTANGULAR OR EQUILATERAL HYPERBOLA :

The particular kind of hyperbola in which the lengths of the transverse & conjugate axis are equal is called an **Equilateral Hyperbola**. Note that the eccentricity of the rectangular hyperbola is $\sqrt{2}$ and the length of its latus rectum is equal to its transverse or conjugate axis.

5. AUXILIARY CIRCLE :www.MathsBySuhag.com , www.TekoClasses.com

A circle drawn with centre C & T.A. as a diameter is called the **Auxiliary Circle** of the hyperbola. Equation of the auxiliary circle is $x^2 + y^2 = a^2$.

Note from the figure that P & Q are called the **"Corresponding Points "** on the hyperbola & the auxiliary circle. 'θ' is called the eccentric angle of the point 'P' on the hyperbola. $(0 \le \theta < 2\pi)$.

Note : The equations $x = a \sec\theta$ & $y = b \tan\theta$ together represents the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ where θ is a parameter. The parametric equations : $x = a \cosh\phi$, $y = b \sinh\phi$ also represents the same hyperbola.

General Note : Since the fundamental equation to the hyperbola only differs from that to the ellipse in having $-b^2$ instead of b^2 it will be found that many propositions for the hyperbola are derived from those for the ellipse by simply changing the sign of b^2.

6. POSITION OF A POINT 'P' w.r.t. A HYPERBOLA :

The quantity $\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} = 1$ is positive, zero or negative according as the point (x_1, y_1) lies within, upon or without the curve.

7. LINE AND A HYPERBOLA : The straight line y = mx + c is a secant, a tangent or passes outside the hyperbola $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ according as: $c^2 > = < a^2 m^2 - b^2$.

8. TANGENTS AND NORMALS : TANGENTS :

(a) Equation of the tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point (x_1, y_1) is $\frac{x x_1}{a^2} - \frac{y y_1}{b^2} = 1$.

<u>**Note:**</u> In general two tangents can be drawn from an external point $(x_1 y_1)$ to the hyperbola and they are $y - y_1 = m_1(x$

(a) The equation of the normal to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at the point $P(x_1, y_1)$ on it is $\frac{a^2x}{x_1}+\frac{b^2y}{y_1}=a^2-b^2$ $=a^2e^2$.

(b) The equation of the normal at the point P (a secθ, b tanθ) on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $\frac{a\,x}{\sec\theta}+\frac{by}{\tan\theta}=a^2+b^2=a^2e^2$.

(c) Equation to the chord of contact, polar, chord with a given middle point, pair of tangents from an external point is to be interpreted as in ellipse.www.MathsBySuhag.com , www.TekoClasses.com

9. DIRECTOR CIRCLE :

The locus of the intersection of tangents which are at right angles is known as the **DIRECTOR CIRCLE** of the hyperbola. The equation to the director circle is : $x^2+y^2=a^2-b^2$.

If $b^2 < a^2$ this circle is real; if $b^2 = a^2$ the radius of the circle is zero & it reduces to a point circle at the origin. In this case the centre is the only point from which the tangents at right angles can be drawn to the curve. If $b^2 > a^2$, the radius of the circle is imaginary, so that there is no such circle & so no tangents at right angle can be drawn to the curve.

10. HIGHLIGHTS ON TANGENT AND NORMAL :

H–1 Locus of the feet of the perpendicular drawn from focus of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ upon any tangent is its auxiliary circle i.e. $x^2+y^2=a^2$ & the product of the feet of these perpendiculars is $b^2\cdot$ (semi C ·A)^2

H–2 The portion of the tangent between the point of contact & the directrix subtends a right angle at the corresponding focus.

H–3 The tangent & normal at any point of a hyperbola bisect the angle between the focal radii. This spells the reflection property of the hyperbola as **"An incoming light ray"** aimed towards one focus is reflected from the outer surface of the hyperbola towards the other focus. It follows that if an ellipse and a hyperbola have the same foci, they cut at right angles at any of their common point.

Note that the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the hyperbola $\frac{x^2}{a^2-k^2}-\frac{y^2}{k^2-b^2}=1\,(a>k>b>0)$ Xare confocal and therefore orthogonal.

H–4 The foci of the hyperbola and the points P and Q in which any tangent meets the tangents at the vertices are concyclic with PQ as diameter of the circle.

When b = a the asymptotes of the rectangular hyperbola. $x^2-y^2=a^2$ are, $y=\pm\, x$ which are at right angles.

Note : **(i)** Equilateral hyperbola ⇔ rectangular hyperbola.

(ii) If a hyperbola is equilateral then the conjugate hyperbola is also equilateral.

(iii) A hyperbola and its conjugate have the same asymptote.

(iv) The equation of the pair of asymptotes differ the hyperbola & the conjugate hyperbola by the same constant only.

(v) The asymptotes pass through the centre of the hyperbola & the bisectors of the angles between the asymptotes are the axes of the hyperbola.www.MathsBySuhag.com , www.TekoClasses.com

(vi) The asymptotes of a hyperbola are the diagonals of the rectangle formed by the lines drawn through the extremities of each axis parallel to the other axis.

(vii) Asymptotes are the tangent to the hyperbola from the centre.

(viii) A simple method to find the coordinates of the centre of the hyperbola expressed as a general equation of degree 2 should be remembered as:

Let f (x, y) = 0 represents a hyperbola.

Find $\frac{\partial f}{\partial x}$ & $\frac{\partial f}{\partial y}$. Then the point of intersection of $\frac{\partial f}{\partial x}=0$ & $\frac{\partial f}{\partial y}=0$ gives the centre of the hyperbola.

12. HIGHLIGHTS ON ASYMPTOTES:

H–1 If from any point on the asymptote a straight line be drawn perpendicular to the transverse axis, the product of the segments of this line, intercepted between the point & the curve is always equal to the square of the semi conjugate axis.

H–2 Perpendicular from the foci on either asymptote meet it in the same points as the corresponding directrix & the common points of intersection lie on the auxiliary circle.

H–3 The tangent at any point P on a hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ with centre C, meets the asymptotes in Q and R and cuts off a Δ CQR of constant area equal to ab from the asymptotes & the portion of the tangent intercepted between the asymptote is bisected at the point of contact. This implies that locus of the centre of the circle circumscribing the Δ CQR in case of a rectangular hyperbola is the hyperbola itself & for a standard hyperbola the locus would be the curve, $4(a^2x^2-b^2y^2)=(a^2+b^2)^2$.

H–4 If the angle between the asymptote of a hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is 2θ then e = secθ.

13.RECTANGULAR HYPERBOLA : Rectangular hyperbola referred to its asymptotesas axis of coordinates.**(a)**Eq. is $xy=c^2$ with parametric representation $x=ct$, $y=c/t$, $t\in R-\{0\}$.

the index .**(ii)** The sum of the indices of x & y in each term is n **(iii)** The binomial coefficients of the terms ${}^nC_0$, ${}^nC_1$ .... **equidistant** from the beginning and the end are equal.

2. IMPORTANT TERMS IN THE BINOMIAL EXPANSION ARE :

(i) General term **(ii)** Middle term **(iii)** Term independent of x & **(iv)** Numerically greatest term

(i) The general term or the $(r + 1)^{th}$ term in the expansion of $(x + y)^n$ is given by ;
$T_{r+1} = {}^nC_r\, x^{n-r} . y^r$

(ii) The middle term(s) is the expansion of $(x + y)^n$ is (are) :

(a) If n is even, there is only one middle term which is given by ;
$T_{(n+2)/2} = {}^nC_{n/2} . x^{n/2} . y^{n/2}$

(b) If n is odd, there are two middle terms which are :
$T_{(n+1)/2}$ & $T_{[(n+1)/2]+1}$

(iii) Term independent of x contains no x ; Hence find the value of r for which the exponent of x is zero.

(iv) To find the Numerically greatest term is the expansion of $(1 + x)^n$, $n \in N$ find $\frac{T_{r+1}}{T_r} = \frac{{}^nC_r x^r}{{}^nC_{r-1} x^{r-1}} = \frac{n-r+1}{r} x$. Put the absolute value of x & find the value of r Consistent with the inequality $\frac{T_{r+1}}{T_r} > 1$.

Note that the Numerically greatest term in the expansion of $(1 - x)^n$, $x > 0$, $n \in N$ is the same as the greatest term in $(1 + x)^n$.

3. If $\left(\sqrt{A}+B\right)^n = I + f$, where I & n are positive integers, n being odd and $0 < f < 1$, then $(I + f) . f = K^n$ where $A - B^2 = K > 0$ & $\sqrt{A} - B < 1$.
If n is an even integer, then $(I + f)(1 - f) = K^n$.

4. BINOMIAL COEFFICIENTS : (i) $C_0 + C_1 + C_2 + ... + C_n = 2^n$

(ii) $C_0 + C_2 + C_4 + ... = C_1 + C_3 + C_5 + ... = 2^{n-1}$

(iii) $C_0^2 + C_1^2 + C_2^2 + ... + C_n^2 = {}^{2n}C_n = \frac{(2n)!}{n!\, n!}$

(iv) $C_0.C_r + C_1.C_{r+1} + C_2.C_{r+2} + ... + C_{n-r}.C_n = \frac{(2n)!}{(n+r)(n-r)!}$

REMEMBER : (i) $(2n)! = 2^n . n! [1. 3. 5 ...... (2n - 1)]$

5. BINOMIAL THEOREM FOR NEGATIVE OR FRACTIONAL INDICES

If $n \in Q$, then $(1+x)^n = 1 + n\,x + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + ...\infty$ Provided $| x | < 1$.

Note : (i) When the index n is a positive integer the number of terms in the expansion of $(1 + x)^n$ is finite i.e. $(n + 1)$ & the coefficient of successive terms are :
${}^nC_0, {}^nC_1, {}^nC_2, {}^nC_3 ..... {}^nC_n$

(ii) When the index is other than a positive integer such as negative integer or fraction, the number of terms in the expansion of $(1 + x)^n$ is infinite and the symbol ${}^nC_r$ cannot be used to denote the

Note : (a) $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + ...\infty$

(b) e is an irrational number lying between 2.7 & 2.8. Its value correct upto 10 places of decimal is 2.7182818284.

(c) $e + e^{-1} = 2\left(1 + \frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + ...\infty\right)$ **(d)** $e - e^{-1} = 2\left(1 + \frac{1}{3!} + \frac{1}{5!} + \frac{1}{7!} + ...\infty\right)$

(e) Logarithms to the base 'e' are known as the Napierian system, so named after Napier, their inventor. They are also called **Natural Logarithm**.

8. LOGARITHMIC SERIES :

(i) $ln\,(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ... \infty$ where $-1 < x \le 1$

(ii) $ln\,(1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} + ... \infty$ where $-1 \le x < 1$

(iii) $ln\,\frac{(1+x)}{(1-x)} = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + ...\infty\right)$ $|x| < 1$

REMEMBER : (a) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ... \infty = ln\,2$ **(b)** $e^{ln\,x} = x$

(c) $ln2 = 0.693$ **(d)** $ln10 = 2.303$

21. VECTOR & 3-D

1. DEFINITIONS:

A VECTOR may be described as a quantity having both magnitude & direction. A vector is generally represented by a directed line segment, say $\vec{AB}$. A is called the **initial point** & B is called the **terminal point**. The magnitude of vector $\vec{AB}$ is expressed by $|\vec{AB}|$.

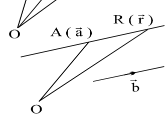
ZERO VECTOR a vector of zero magnitude i.e.which has the same initial & terminal point, is called a **ZERO VECTOR**. It is denoted by O.

UNIT VECTOR a vector of unit magnitude in direction of a vector $\vec{a}$ is called unit vector along $\vec{a}$ and is denoted by $\hat{a}$ symbolically $\hat{a} = \frac{\vec{a}}{|\vec{a}|}$. **EQUAL VECTORS** two vectors are said to be equal if they have the same magnitude, direction & represent the same physical quantity. **COLLINEAR VECTORS** two vectors are said to be collinear if their directed line segments are parallel disregards to their direction. Collinear vectors are also called **PARALLEL VECTORS**. If they have the same direction they are named as like vectors otherwise

$m(\vec{a})=(\vec{a})m=m\vec{a}$

$m(n\vec{a})=n(m\vec{a})=(mn)\vec{a}$

$(m+n)\vec{a}=m\vec{a}+n\vec{a}$ $\qquad m(\vec{a}+\vec{b})=m\vec{a}+m\vec{b}$



4. SECTION FORMULA :

If $\vec{a}$ & $\vec{b}$ are the position vectors of two points A & B then the p.v. of a point which divides AB in the ratio m : n is given by : $\vec{r}=\dfrac{n\vec{a}+m\vec{b}}{m+n}$. Note p.v. of mid point of AB = $\dfrac{\vec{a}+\vec{b}}{2}$

5. DIRECTION COSINES : www.MathsBySuhag.com , www.TekoClasses.com

Let $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ the angles which this vector makes with the +ve directions OX,OY & OZ are called **DIRECTION ANGLES** & their cosines are called the **DIRECTION COSINES** . $\cos\alpha=\dfrac{a_1}{|\vec{a}|}$, $\cos\beta=\dfrac{a_2}{|\vec{a}|}$, $\cos\Gamma=\dfrac{a_3}{|\vec{a}|}$. **Note that, $\cos^2\alpha+\cos^2\beta+\cos^2\Gamma=1$**

6. VECTOR EQUATION OF A LINE :

Parametric vector equation of a line passing through two point $A(\vec{a})$ & $B(\vec{b})$ is given by, $\vec{r}=\vec{a}+t(\vec{b}-\vec{a})$ where t is a parameter. If the line passes through the point $A(\vec{a})$ & is parallel to the vector $\vec{b}$ then its equation is, $\vec{r}=\vec{a}+t\vec{b}$

Note that the equations of the bisectors of the angles between the lines $\vec{r}=\vec{a}+\lambda\vec{b}$ & $\vec{r}=\vec{a}+\mu\vec{c}$ is : $\vec{r}=\vec{a}+t\left(\hat{b}+\hat{c}\right)$ & $\vec{r}=\vec{a}+p\left(\hat{c}-\hat{b}\right)$.

7. TEST OF COLLINEARITY :

Three points A,B,C with position vectors $\vec{a},\vec{b},\vec{c}$ respectively are collinear, if & only if there exist scalars x , y , z not all zero simultaneously such that ; $x\vec{a}+y\vec{b}+z\vec{c}=0$, where x + y + z = 0.

8. SCALAR PRODUCT OF TWO VECTORS :

☞ $\vec{a}.\vec{b}=|\vec{a}||\vec{b}|\cos\theta(0\le\theta\le\pi)$, note that if θ is acute then $\vec{a}.\vec{b}>0$ & if θ is obtuse then $\vec{a}.\vec{b}<0$ ☞

Note : (i) Maximum value of $\vec{a}\,.\,\vec{b}=|\vec{a}|\,|\vec{b}|$

(ii) Minimum values of $\vec{a}\,.\,\vec{b}=\vec{a}\,.\,\vec{b}=-|\vec{a}|\,|\vec{b}|$

(iii) Any vector $\vec{a}$ can be written as , $\vec{a}=\left(\vec{a}\,.\,\hat{i}\right)\hat{i}+\left(\vec{a}\,.\,\hat{j}\right)\hat{j}+\left(\vec{a}\,.\,\hat{k}\right)\hat{k}$.

(iv) A vector in the direction of the bisector of the angle between the two vectors $\vec{a}\,\&\,\vec{b}$ is $\dfrac{\vec{a}}{|\vec{a}|}+\dfrac{\vec{b}}{|\vec{b}|}$. Hence bisector of the angle between the two vectors $\vec{a}\,\&\,\vec{b}$ is $\lambda\left(\hat{a}+\hat{b}\right)$, where $\lambda\in R^+$. Bisector of the exterior angle between $\vec{a}\,\&\,\vec{b}$ is $\lambda\left(\hat{a}-\hat{b}\right)$, $\lambda\in R^+$.

9. VECTOR PRODUCT OF TWO VECTORS :

(i) ☞ If $\vec{a}\,\&\,\vec{b}$ are two vectors & θ is the angle between them then $\vec{a}\times\vec{b}=|\vec{a}||\vec{b}|\sin\theta\vec{n}$, where $\vec{n}$ is the unit vector perpendicular to both $\vec{a}\,\&\,\vec{b}$ such that $\vec{a},\vec{b}\,\&\,\vec{n}$ forms a right handed screw system .

(ii) ☞ Lagranges Identity : for any two vectors $\vec{a}$ & $\vec{b}$; $(\vec{a}\times\vec{b})^2=|\vec{a}|^2|\vec{b}|^2-(\vec{a}.\vec{b})^2=\begin{vmatrix}\vec{a}.\vec{a} & \vec{a}.\vec{b}\\ \vec{a}.\vec{b} & \vec{b}.\vec{b}\end{vmatrix}$

(iii) ☞ Formulation of vector product in terms of scalar product:

The vector product $\vec{a}\times\vec{b}$ is the vector $\vec{c}$, such that

(i) $|\vec{c}|=\sqrt{\vec{a}^2\vec{b}^2-(\vec{a}\cdot\vec{b})^2}$ (ii) $\vec{c}\cdot\vec{a}=0$; $\vec{c}\cdot\vec{b}=0$ and (iii) $\vec{a},\vec{b},\vec{c}$ form a right handed system

(iv) ☞ $\vec{a}\times\vec{b}=0\Leftrightarrow\vec{a}\,\&\,\vec{b}$ are parallel (collinear) $(\vec{a}\ne0,\vec{b}\ne0)$ i.e. $\vec{a}=K\vec{b}$, where K is a scalar.

☞ $\vec{a}\times\vec{b}\ne\vec{b}\times\vec{a}$ (not commutative)

☞ $(m\vec{a})\times\vec{b}=\vec{a}\times(m\vec{b})=m(\vec{a}\times\vec{b})$ where m is a scalar .

☞ $\vec{a}\times(\vec{b}+\vec{c})=(\vec{a}\times\vec{b})+(\vec{a}\times\vec{c})$ (distributive)

☞ $\hat{i}\times\hat{i}=\hat{j}\times\hat{j}=\hat{k}\times\hat{k}=0$ ☞ $\hat{i}\times\hat{j}=\hat{k},\ \hat{j}\times\hat{k}=\hat{i},\ \hat{k}\times\hat{i}=\hat{j}$

(v) ☞ If $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ & $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ then $\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$

If two lines in space intersect at a point, then obviously the shortest distance between them is zero. Lines which do not intersect & are also not parallel are called **SKEW LINES**. For Skew lines the direction of the shortest distance would be perpendicular to both the lines. The magnitude of the shortest distance vector would be equal to that of the projection of $\vec{AB}$ along the direction of the line of shortest distance, $\vec{LM}$ is parallel to $\vec{p}\times\vec{q}$ i.e. $\vec{LM} = \left|\text{Projection of } \vec{AB} \text{ on } \vec{LM}\right| = \left|\text{Projection of } \vec{AB} \text{ on } \vec{p}\times\vec{q}\right|$

$= \left|\dfrac{\vec{AB}\cdot(\vec{p}\times\vec{q})}{\vec{p}\times\vec{q}}\right| = \left|\dfrac{(\vec{b}-\vec{a})\cdot(\vec{p}\times\vec{q})}{|\vec{p}\times\vec{q}|}\right|$ www.MathsBySuhag.com , www.TekoClasses.com

1. The two lines directed along $\vec{p}$ & $\vec{q}$ will intersect only if shortest distance = 0 i.e.

$(\vec{b}-\vec{a})\cdot(\vec{p}\times\vec{q})=0$ i.e. $(\vec{b}-\vec{a})$ lies in the plane containing $\vec{p}$ & $\vec{q}$. $\Rightarrow \left[(\vec{b}-\vec{a})\ \vec{p}\ \vec{q}\right]=0$

2. If two lines are given by $\vec{r}_1=\vec{a}_1+K\vec{b}$ & $\vec{r}_2=\vec{a}_2+K\vec{b}$ i.e. they are parallel then , $d=\left|\dfrac{\vec{b}\times(\vec{a}_2-\vec{a}_1)}{|\vec{b}|}\right|$

11. SCALAR TRIPLE PRODUCT / BOX PRODUCT / MIXED PRODUCT :

☞ The scalar triple product of three vectors $\vec{a},\vec{b}\,\&\,\vec{c}$ is defined as :

$\vec{a}\times\vec{b}\cdot\vec{c}=|\vec{a}||\vec{b}||\vec{c}|\ \sin\theta\ \cos\phi$ where θ is the angle between $\vec{a}$ & $\vec{b}$ & ϕ is the angle between $\vec{a}\times\vec{b}$ & $\vec{c}$. It is also defined as $[\vec{a}\vec{b}\vec{c}]$, spelled as box product .

☞ Scalar triple product geometrically represents the volume of the parallelopiped whose three couterminous edges are represented by $\vec{a},\vec{b}\,\&\,\vec{c}$ i.e. $V=[\vec{a}\vec{b}\vec{c}]$

☞ In a scalar triple product the position of dot & cross can be interchanged i.e.

$\vec{a}\cdot(\vec{b}\times\vec{c})=(\vec{a}\times\vec{b})\cdot\vec{c}$ OR $[\vec{a}\vec{b}\vec{c}]=[\vec{b}\vec{c}\vec{a}]=[\vec{c}\vec{a}\vec{b}]$

☞ $\vec{a}\cdot(\vec{b}\times\vec{c})=-\vec{a}\cdot(\vec{c}\times\vec{b})$ i.e. $[\vec{a}\ \vec{b}\ \vec{c}] = -[\vec{a}\ \vec{c}\ \vec{b}]$

☞ If $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$; $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ & $\vec{c}=c_1\hat{i}+c_2\hat{j}+c_3\hat{k}$ then .

In general , if $\vec{a}=a_1\vec{l}+a_2\vec{m}+a_3\vec{n}$; $\vec{b}=b_1\vec{l}+b_2\vec{m}+b_3\vec{n}$ & $\vec{c}=c_1\vec{l}+c_2\vec{m}+c_3\vec{n}$

Note that this is also the point of concurrency of the lines joining the vertices to the centroids of the opposite faces and is also called the centre of the tetrahedron. In case the tetrahedron is regular it is equidistant from the vertices and the four faces of the tetrahedron .

Remember that : $\left[\vec{a}-\vec{b}\quad \vec{b}-\vec{c}\quad \vec{c}-\vec{a}\right]=0$ & $\left[\vec{a}+\vec{b}\quad \vec{b}+\vec{c}\quad \vec{c}+\vec{a}\right]=2\left[\vec{a}\ \vec{b}\ \vec{c}\right]$.

***12. VECTOR TRIPLE PRODUCT :**

Let $\vec{a},\vec{b},\vec{c}$ be any three vectors, then the expression $\vec{a}\times(\vec{b}\times\vec{c})$ is a vector & is called a vector triple product .www.MathsBySuhag.com , www.TekoClasses.com

GEOMETRICAL INTERPRETATION OF $\vec{a}\times(\vec{b}\times\vec{c})$

Consider the expression $\vec{a}\times(\vec{b}\times\vec{c})$ which itself is a vector, since it is a cross product of two vectors $\vec{a}$ & $(\vec{b}\times\vec{c})$. Now $\vec{a}\times(\vec{b}\times\vec{c})$ is a vector perpendicular to the plane containing $\vec{a}$ & $(\vec{b}\times\vec{c})$ but $\vec{b}\times\vec{c}$ is a vector perpendicular to the plane $\vec{b}\,\&\,\vec{c}$, therefore $\vec{a}\times(\vec{b}\times\vec{c})$ is a vector lies in the plane of $\vec{b}\,\&\,\vec{c}$ and perpendicular to $\vec{a}$. Hence we can express $\vec{a}\times(\vec{b}\times\vec{c})$ in terms of $\vec{b}\,\&\,\vec{c}$ i.e. $\vec{a}\times(\vec{b}\times\vec{c}) = x\vec{b}+y\vec{c}$ where x & y are scalars .

☞ $\vec{a}\times(\vec{b}\times\vec{c})=(\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}$ ☞ $(\vec{a}\times\vec{b})\times\vec{c}=(\vec{a}\cdot\vec{c})\vec{b}-(\vec{b}\cdot\vec{c})\vec{a}$

☞ $(\vec{a}\times\vec{b})\times\vec{c}\neq\vec{a}\times(\vec{b}\times\vec{c})$

13. LINEAR COMBINATIONS / Linearly Independence and Dependence of Vectors :

Given a finite set of vectors $\vec{a},\vec{b},\vec{c},......$ then the vector $\vec{r}=x\vec{a}+y\vec{b}+z\vec{c}+........$ is called a linear combination of $\vec{a},\vec{b},\vec{c},......$ for any x, y, z ...... $\in$ R. We have the following results :

(a) **FUNDAMENTAL THEOREM IN PLANE :** Let $\vec{a},\vec{b}$ be non zero , non collinear vectors . Then any vector $\vec{r}$ coplanar with $\vec{a},\vec{b}$ can be expressed uniquely as a linear combination of $\vec{a},\vec{b}$ i.e. There exist some unique $x,y\in$ R such that $x\vec{a}+y\vec{b}=\vec{r}$.

(b) **FUNDAMENTAL THEOREM IN SPACE :** Let $\vec{a},\vec{b},\vec{c}$ be non–zero, non–coplanar vectors in space. Then any vector $\vec{r}$, can be uniquily expressed as a linear combination of $\vec{a},\vec{b},\vec{c}$ i.e. There exist some unique x,y $\in$ R such that $x\vec{a}+y\vec{b}+z\vec{c}=\vec{r}$.

(c) If $\vec{x}_1,\vec{x}_2,......\vec{x}_n$ are n non zero vectors, & $k_1, k_2, ...k_n$ are n scalars & if the linear combination

14. COPLANARITY OF VECTORS :

Four points A, B, C, D with position vectors $\vec{a},\vec{b},\vec{c},\vec{d}$ respectively are coplanar if and only if there exist scalars x, y, z, w not all zero simultaneously such that $x\vec{a}+y\vec{b}+z\vec{c}+w\vec{d}=0$ where, x + y + z + w = 0.

15. RECIPROCAL SYSTEM OF VECTORS :

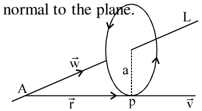
If $\vec{a},\vec{b},\vec{c}$ & $\vec{a}',\vec{b}',\vec{c}'$ are two sets of non coplanar vectors such that $\vec{a}.\vec{a}'=\vec{b}.\vec{b}'=\vec{c}.\vec{c}'=1$ then the two systems are called Reciprocal System of vectors. **Note :** $a'=\frac{\vec{b}\times\vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]}$; $b'=\frac{\vec{c}\times\vec{a}}{[\vec{a}\,\vec{b}\,\vec{c}]}$; $c'=\frac{\vec{a}\times\vec{b}}{[\vec{a}\,\vec{b}\,\vec{c}]}$

16. EQUATION OF A PLANE :www.MathsBySuhag.com , www.TekoClasses.com

(a) The equation $(\vec{r}-\vec{r}_0).\vec{n}=0$ represents a plane containing the point with p.v. $\vec{r}_0$ where $\vec{n}$ is a vector normal to the plane . $\vec{r}.\vec{n}=d$ is the general equation of a plane.

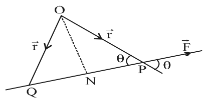
(b) Angle between the 2 planes is the angle between 2 normals drawn to the planes and the angle between a line and a plane is the compliment of the angle between the line and the normal to the plane.

17. APPLICATION OF VECTORS :



(a) Work done against a constant force $\vec{F}$ over a displacement $\vec{s}$ is defined as $\vec{W}=\vec{F}.\vec{s}$

(b) The tangential velocity $\vec{V}$ of a body moving in a circle is given by $\vec{V}=\vec{w}\times\vec{r}$ where $\vec{r}$ is the pv of the point P.



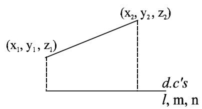
(c) The moment of $\vec{F}$ about 'O' is defined as $\vec{M}=\vec{r}\times\vec{F}$ where $\vec{r}$ is the pv of P wrt 'O'. The direction of $\vec{M}$ is along the normal to the plane OPN such that $\vec{r},\vec{F}$ & $\vec{M}$ form a right handed system.

(d) Moment of the couple $=(\vec{r}_1-\vec{r}_2)\times\vec{F}$ where $\vec{r}_1$ & $\vec{r}_2$ are pv's of the point of the application of the forces $\vec{F}$ & $-\vec{F}$.

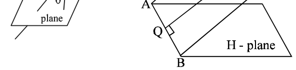
3 -D COORDINATE GEOMETRY

USEFUL RESULTS

A General :



(1) Distance (d) between two points (x_1, y_1, z_1) and (x_2, y_2, z_2)



note that if three lines are coplanar then $\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}=0$

(4)Projection of join of 2 points on line with d.c's l, m, n are $l(x_2-x_1)+m(y_2-y_1)+n(z_2-z_1)$

B PLANEwww.MathsBySuhag.com , www.TekoClasses.com

(i) General equation of degree one in x, y, z i.e. ax + by + cz + d = 0 represents a plane.

(ii) Equation of a plane passing through (x_1, y_1, z_1) is
$a(x-x_1)+b(y-y_1)+c(z-z_1)=0$ where a, b, c are the direction ratios of the normal to the plane.

(iii) Equation of a plane if its intercepts on the co-ordinate axes are x_1, y_1, z_1 is $\frac{x}{x_1}+\frac{y}{y_1}+\frac{z}{z_1}=1$.

(iv) Equation of a plane if the length of the perpendicular from the origin on the plane is p and d.c's of the perpendicular as l, m, , n is $l x+m y+n z=p$

(v) **Parallel and perpendicular planes** – Two planes $a_1x+b_1y+c_1z+d_1=0$ and $a_2x+b_2y+c_2z+d_2=0$ are perpendicular if $a_1a_2+b_1b_2+c_1c_2=0$

parallel if $\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$ and coincident if $\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}=\frac{d_1}{d_2}$

(vi) Angle between a plane and a line is the compliment of the angle between the normal to the plane and the line . If $\left.\begin{array}{l}\text{Line} : \vec{r}=\vec{a}+\lambda\vec{b} \\ \text{Plane} : \vec{r}.\vec{n}=d\end{array}\right]$ then $\cos(90-\theta)=\sin\theta=\frac{\vec{b}.\vec{n}}{|\vec{b}|.|\vec{n}|}$.

where θ is the angle between the line and normal to the plane.

(vii) Length of the perpendicular from a point (x_1, y_1, z_1) to a plane ax + by + cz + d = 0 is

$$p=\left|\frac{ax_1+by_1+cz_1+d}{\sqrt{a^2+b^2+c^2}}\right|$$

(viii) Distance between two parallel planes $ax+by+cz+d_1=0$ and $ax+by+cz+d_2=0$ is

$$\left|\frac{d_1-d_2}{\sqrt{a^2+b^2+c^2}}\right|$$

(ix) Planes bisecting the angle between two planes $a_1x+b_1y+c_1z+d_1=0$ and $a_2+b_2y+c_2z+d_2=0$ is

AB is the line of intersection of G-plane and H is the horizontal plane. Line of greatest slope on a given plane, drawn through a given point on the plane, is the line through the point 'P' perpendicular to the line of intersetion of the given plane with any horizontal plane.

22. TRIGONOMETRY-1 (COMPOUND ANGLE)

1. **Basic Trigonometric Identities :**

(a) $\sin^2\theta + \cos^2\theta = 1$; $-1 \le \sin\theta \le 1$; $-1 \le \cos\theta \le 1$ $\forall$ $\theta \in R$

(b) $\sec^2\theta - \tan^2\theta = 1$; $|\sec\theta| \ge 1$ $\forall$ $\theta \in R$

(c) $\text{cosec}^2\theta - \cot^2\theta = 1$; $|\text{cosec}\,\theta| \ge 1$ $\forall$ $\theta \in R$

2. **Important T' Ratios:**

(a) $\sin n\pi = 0$; $\cos n\pi = (-1)^n$; $\tan n\pi = 0$ where $n \in I$

(b) $\sin\frac{(2n+1)\pi}{2} = (-1)^n$ & $\cos\frac{(2n+1)\pi}{2} = 0$ where $n \in I$

(c) $\sin 15°$ or $\sin\frac{\pi}{12} = \frac{\sqrt{3}-1}{2\sqrt{2}} = \cos 75°$ or $\cos\frac{5\pi}{12}$;

$\cos 15°$ or $\cos\frac{\pi}{12} = \frac{\sqrt{3}+1}{2\sqrt{2}} = \sin 75°$ or $\sin\frac{5\pi}{12}$;

$\tan 15° = \frac{\sqrt{3}-1}{\sqrt{3}+1} = 2-\sqrt{3} = \cot 75°$; $\tan 75° = \frac{\sqrt{3}+1}{\sqrt{3}-1} = 2+\sqrt{3} = \cot 15°$

(d) $\sin\frac{\pi}{8} = \frac{\sqrt{2-\sqrt{2}}}{2}$; $\cos\frac{\pi}{8} = \frac{\sqrt{2+\sqrt{2}}}{2}$; $\tan\frac{\pi}{8} = \sqrt{2}-1$; $\tan\frac{3\pi}{8} = \sqrt{2}+1$

(e) $\sin\frac{\pi}{10}$ or $\sin 18° = \frac{\sqrt{5}-1}{4}$ & $\cos 36°$ or $\cos\frac{\pi}{5} = \frac{\sqrt{5}+1}{4}$

3. **Trigonometric Functions Of Allied Angles :**

If θ is any angle, then $-\theta$, $90 \pm \theta$, $180 \pm \theta$, $270 \pm \theta$, $360 \pm \theta$ etc. are called **Allied Angles**.

(a) $\sin(-\theta) = -\sin\theta$; $\cos(-\theta) = \cos\theta$

(b) $\sin(90°-\theta) = \cos\theta$; $\cos(90°-\theta) = \sin\theta$

(c) $\sin(90°+\theta) = \cos\theta$; $\cos(90°+\theta) = -\sin\theta$

(d) $\sin(180°-\theta) = \sin\theta$; $\cos(180°-\theta) = -\cos\theta$

(c) $2\cos A\cos B = \cos(A+B) + \cos(A-B)$ **(d)** $2\sin A\sin B = \cos(A-B) - \cos(A+B)$

7. **Multiple Angles And Half Angles :**www.MathsBySuhag.com , www.TekoClasses.com

(a) $\sin 2A = 2\sin A\cos A$; $\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$

(b) $\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$;

$\cos\theta = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} = 2\cos^2\frac{\theta}{2} - 1 = 1 - 2\sin^2\frac{\theta}{2}$.

$2\cos^2 A = 1 + \cos 2A$, $2\sin^2 A = 1 - \cos 2A$; $\tan^2 A = \frac{1-\cos 2A}{1+\cos 2A}$

$2\cos^2\frac{\theta}{2} = 1 + \cos\theta$, $2\sin^2\frac{\theta}{2} = 1 - \cos\theta$.

(c) $\tan 2A = \frac{2\tan A}{1-\tan^2 A}$; $\tan\theta = \frac{2\tan(\theta/2)}{1-\tan^2(\theta/2)}$

(d) $\sin 2A = \frac{2\tan A}{1+\tan^2 A}$, $\cos 2A = \frac{1-\tan^2 A}{1+\tan^2 A}$ **(e)** $\sin 3A = 3\sin A - 4\sin^3 A$

(f) $\cos 3A = 4\cos^3 A - 3\cos A$ **(g)** $\tan 3A = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$

8. **Three Angles :**

(a) $\tan(A+B+C) = \frac{\tan A + \tan B + \tan C - \tan A\tan B\tan C}{1 - \tan A\tan B - \tan B\tan C - \tan C\tan A}$

Note If : **(i)** $A+B+C = \pi$ then $\tan A + \tan B + \tan C = \tan A\tan B\tan C$

(ii) $A+B+C = \frac{\pi}{2}$ then $\tan A\tan B + \tan B\tan C + \tan C\tan A = 1$

(b) If $A + B + C = \pi$ then : **(i)** $\sin 2A + \sin 2B + \sin 2C = 4\sin A\sin B\sin C$

(ii) $\sin A + \sin B + \sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

9. **Maximum & Minimum values of Trigonometric Functions:**

(a) Min. value of $a^2\tan^2\theta + b^2\cot^2\theta = 2ab$ where $\theta \in R$

(b) Max. and Min. value of $a\cos\theta + b\sin\theta$ are $\sqrt{a^2+b^2}$ and $-\sqrt{a^2+b^2}$

(c) If $f(\theta) = a\cos(\alpha+\theta) + b\cos(\beta+\theta)$ where a, b, α and β are known quantities then $-\sqrt{a^2+b^2+2ab\cos(\alpha-\beta)} \le f(\theta) \le \sqrt{a^2+b^2+2ab\cos(\alpha-\beta)}$

(d) If $\alpha,\beta \in \left(0,\frac{\pi}{2}\right)$ and $\alpha + \beta = \sigma$ (constant) then the maximum values of the expression $\cos\alpha\cos\beta$, $\cos\alpha + \cos\beta$, $\sin\alpha + \sin\beta$ and $\sin\alpha\sin\beta$ occurs when $\alpha = \beta = \sigma/2$.

(e) If $\alpha,\beta \in \left(0,\frac{\pi}{2}\right)$ and $\alpha + \beta = \sigma$(constant) then the minimum values of the expression $\sec\alpha + \sec\beta$,

2. If $\cos\theta = \cos\alpha \Rightarrow \theta = 2n\pi \pm \alpha$ where $\alpha \in [0, \pi]$, $n \in I$.

3. If $\tan\theta = \tan\alpha \Rightarrow \theta = n\pi + \alpha$ where $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $n \in I$.

4. If $\sin^2\theta = \sin^2\alpha \Rightarrow \theta = n\pi \pm \alpha$.

5. $\cos^2\theta = \cos^2\alpha \Rightarrow \theta = n\pi \pm \alpha$.

6. $\tan^2\theta = \tan^2\alpha \Rightarrow \theta = n\pi \pm \alpha$. [Note : α is called the principal angle]

7. **TYPES OF TRIGONOMETRIC EQUATIONS :**

(a) Solutions of equations by factorising . Consider the equation ;
$(2\sin x - \cos x)(1 + \cos x) = \sin^2 x$; $\cot x - \cos x = 1 - \cot x \cos x$

(b) Solutions of equations reducible to quadratic equations. Consider the equation
$3\cos^2 x - 10\cos x + 3 = 0$ and $2\sin^2 x + \sqrt{3}\sin x + 1 = 0$

(c) Solving equations by introducing an Auxilliary argument . Consider the equation : $\sin x + \cos x = \sqrt{2}$; $\sqrt{3}\cos x + \sin x = 2$; $\sec x - 1 = (\sqrt{2} - 1)\tan x$

(d) Solving equations by Transforming a sum of Trigonometric functions into a product. Consider the example : $\cos 3x + \sin 2x - \sin 4x = 0$;
$\sin^2 x + \sin^2 2x + \sin^2 3x + \sin^2 4x = 2$; $\sin x + \sin 5x = \sin 2x + \sin 4x$

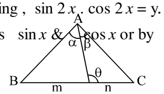
(e) Solving equations by transforming a product of trigonometric functions into a sum. Consider the equation : $\sin 5x \cdot \cos 3x = \sin 6x \cdot \cos 2x$; $8\cos x \cos 2x \cos 4x = \frac{\sin 6x}{\sin x}$**;** $\sin 3\theta = 4\sin\theta \sin 2\theta \sin 4\theta$www.MathsBySuhag.com , www.TekoClasses.com

(f) Solving equations by a change of variable :

(i) Equations of the form of $a \cdot \sin x + b \cdot \cos x + d = 0$, where a, b & d are real numbers & $a, b \neq 0$ can be solved by changing $\sin x$ & $\cos x$ into their corresponding tangent of half the angle. Consider the equation $3\cos x + 4\sin x = 5$.

(ii) Many equations can be solved by introducing a new variable . eg. the equation $\sin^4 2x + \cos^4 2x = \sin 2x \cdot \cos 2x$ changes to $2(y+1)\left(y - \frac{1}{2}\right) = 0$ by substituting , $\sin 2x \cdot \cos 2x = y$.

(g) Solving equations with the use of the Boundness of the functions $\sin x$ & $\cos x$ or by making two perfect squares. Consider the equations :



$$\sin x\left(\cos\frac{x}{4} - 2\sin x\right) + \left(1 + \sin\frac{x}{4} - 2\cos x\right) \cdot \cos x = 0 \;;$$

$$\sin^2 x + 2\tan^2 x + \frac{4}{\sqrt{3}}\tan x - \sin x + \frac{11}{12} = 0$$

(ii) $\tan\frac{C-A}{2} = \frac{c-a}{c+a}\cot\frac{B}{2}$ (iii) $\tan\frac{A-B}{2} = \frac{a-b}{a+b}\cot\frac{C}{2}$

V. TRIGONOMETRIC FUNCTIONS OF HALF ANGLES :

(i) $\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$; $\sin\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{ca}}$; $\sin\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$

(ii) $\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$; $\cos\frac{B}{2} = \sqrt{\frac{s(s-b)}{ca}}$; $\cos\frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}$

(iii) $\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \frac{\Delta}{s(s-a)}$ where $s = \frac{a+b+c}{2}$ & Δ = area of triangle.

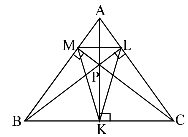
(iv) Area of triangle = $\sqrt{s(s-a)(s-b)(s-c)}$.

VI. M – N RULE : In any triangle ,
$(m+n)\cot\theta = m\cot\alpha - n\cot\beta$
$= n\cot B - m\cot C$

VII. $\frac{1}{2}ab\sin C = \frac{1}{2}bc\sin A = \frac{1}{2}ca\sin B$ = area of triangle ABC .

$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$

Note that $R = \frac{abc}{4\Delta}$; Where R is the radius of circumcircle & Δ is area of trianglewww.MathsBySuhag.com , www.TekoClasses.com



VIII. Radius of the incircle 'r' is given by :

(a) $r = \frac{\Delta}{s}$ where $s = \frac{a+b+c}{2}$ **(b)** $r = (s-a)\tan\frac{A}{2} = (s-b)\tan\frac{B}{2} = (s-c)\tan\frac{C}{2}$

(c) $r = \frac{a\sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{A}{2}}$ & so on **(d)** $r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

IX. Radius of the Ex– circles r_1, r_2 & r_3 are given by :

(a) $r_1 = \frac{\Delta}{s-a}$; $r_2 = \frac{\Delta}{s-b}$; $r_3 = \frac{\Delta}{s-c}$ **(b)** $r_1 = s\tan\frac{A}{2}$; $r_2 = s\tan\frac{B}{2}$; $r_3 = s\tan\frac{C}{2}$

– circumradii of the triangles PBC, PCA, PAB and ABC are equal .

XII **EXCENTRAL TRIANGLE :** The triangle formed by joining the three excentres I_1, I_2 and I_3 of Δ ABC is called the excentral or excentric triangle.

Note that :– Incentre I of Δ ABC is the orthocentre of the excentral $\Delta I_1I_2I_3$.

– Δ ABC is the pedal triangle of the $\Delta I_1I_2I_3$.– the sides of the excentral triangle are

$4R\cos\frac{A}{2}$, $4R\cos\frac{B}{2}$ and $4R\cos\frac{C}{2}$ and its angles are $\frac{\pi}{2}-\frac{A}{2}$, $\frac{\pi}{2}-\frac{B}{2}$ and $\frac{\pi}{2}-\frac{C}{2}$.

– $II_1 = 4R\sin\frac{A}{2}$; $II_2 = 4R\sin\frac{B}{2}$; $II_3 = 4R\sin\frac{C}{2}$.

XIII. **THE DISTANCES BETWEEN THE SPECIAL POINTS :** www.MathsBySuhag.com , www.TekoClasses.com

(a) The distance between circumcentre and orthocentre is $= R\,.\sqrt{1-8\cos A\cos B\cos C}$

(b) The distance between circumcentre and incentre is $= \sqrt{R^2-2Rr}$

(c) The distance between incentre and orthocentre is $\sqrt{2r^2-4R^2\cos A\cos B\cos C}$

XIV. Perimeter (P) and area (A) of a regular polygon of n sides inscribed in a circle of radius r are given by $P = 2nr\sin\frac{\pi}{n}$ and $A = \frac{1}{2}nr^2\sin\frac{2\pi}{n}$ Perimeter and area of a regular polygon of n sides circumscribed about a given circle of radius r is given by $P = 2nr\tan\frac{\pi}{n}$ and $A = nr^2\tan\frac{\pi}{n}$

XV. In many kinds of trignometric calculation, as in the solution of triangles, we often require the logarithms of trignometrical ratios . To avoid the trouble and inconvenience of printing the proper sign to the logarithms of the trignometric functions, the logarithms as tabulated are not the true logarithms, but the true logarithms increased by 10 . The symbol L is used to denote these "tabular logarithms" . Thus :

$$L\sin 15^\circ 25' = 10 + \log_{10}\sin 15^\circ 25' \quad\text{and}\quad L\tan 48^\circ 23' = 10 + \log_{10}\tan 48^\circ 23'$$

IIT JEE ADVANCED Physics Syllabus

General: Units and dimensions, dimensional analysis; least count, significant figures; Methods of measurement and error analysis for physical quantities pertaining to the following experiments: Experiments based on using Vernier calipers and screw gauge (micrometer), Determination of g using simple pendulum, Young?s modulus by Searle?s method, Specific heat of a liquid using calorimeter, focal length of a concave mirror and a convex lens using u-v method, Speed of sound using resonance column, Verification of Ohm?s law using voltmeter and ammeter, and specific resistance of the material of a wire using meter bridge and post office box.

Mechanics: Kinematics in one and two dimensions (Cartesian coordinates only), projectiles; Uniform Circular motion; Relative velocity.

Newton's laws of motion; Inertial and uniformly accelerated frames of reference; Static and dynamic friction;

Specific heats (Cv and Cp for monoatomic and diatomic gases); Isothermal and adiabatic processes, bulk modulus of gases; Equivalence of heat and work; First law of thermodynamics and its applications (only for ideal gases);? Blackbody radiation: absorptive and emissive powers; Kirchhoff?s law; Wien?s displacement law, Stefan?s law.

Electricity and magnetism: Coulomb?s law; Electric field and potential; Electrical potential energy of a system of point charges and of electrical dipoles in a uniform electrostatic field; Electric field lines; Flux of electric field; Gauss?s law and its application in simple cases, such as, to find field due to infinitely long straight wire, uniformly charged infinite plane sheet and uniformly charged thin spherical shell.

Capacitance; Parallel plate capacitor with and without dielectrics; Capacitors in series and parallel; Energy stored in a capacitor.

Electric current; Ohm?s law; Series and parallel arrangements of resistances and cells; Kirchhoff?s laws and simple applications; Heating effect of current.

Biot'Savart's law and Ampere?s law; Magnetic field near a current-carrying straight wire, along the axis of a circular coil and inside a long straight solenoid; Force on a moving charge and on a current-carrying wire in a uniform magnetic field.

Magnetic moment of a current loop; Effect of a uniform magnetic field on a current loop; Moving coil galvanometer, voltmeter, ammeter and their conversions.

Electromagnetic induction: Faraday?s law, Lenz?s law; Self and mutual inductance; RC, LR and LC circuits with D.C. and A.C. sources.

Optics: Rectilinear propagation of light; Reflection and refraction at plane and spherical surfaces; Total internal reflection; Deviation and dispersion of light by a prism; Thin lenses; Combinations of mirrors and thin lenses; Magnification.?

Wave nature of light: Huygen?s principle, interference limited to Young?s double-slit experiment.

Modern physics: Atomic nucleus; Alpha, beta and gamma radiations; Law of radioactive decay;? Decay constant; Half-life and mean life; Binding energy and its calculation; Fission and fusion processes; Energy calculation in these processes.

Photoelectric effect; Bohr?s theory of hydrogen-like atoms; Characteristic and continuous X-rays, Moseley?s law; de Broglie wavelength of matter waves.